

# Proportions of Cyclic Matrices in Maximal Reducible Matrix Groups and Algebras

Scott Brown, Cheryl E. Praeger, Michael Giudici

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## Abstract

A matrix is said to be *cyclic* if its characteristic polynomial is equal to its minimal polynomial. Cyclic matrices play an important role in some algorithms for matrix group computation, such as the CYCLIC MEATAXE developed by P. M. Neumann and C. E. Praeger in 1999. In that year also, G. E. Wall and J. E. Fulman independently found the limiting proportion of cyclic matrices in general linear groups over a finite field of fixed order  $q$  as the dimension  $n$  approaches infinity, namely  $(1 - q^{-5}) \prod_{i=3}^{\infty} (1 - q^{-i}) = 1 - q^{-3} + O(q^{-4})$ . We study cyclic matrices in a maximal reducible matrix group or algebra, that is, in the largest subgroup or subalgebra that leaves invariant some proper nontrivial subspace. We modify Wall's generating function approach to determine the limiting proportions of cyclic matrices in maximal reducible matrix groups and algebras over a field of order  $q$ , as the dimension of the underlying vector space increases while that of the invariant subspace remains fixed. The limiting proportion in a maximal reducible group is proved to be  $1 - q^{-2} + O(q^{-3})$ ; note the change of the exponent of  $q$  in the second term of the expansion. Moreover, we exhibit in each maximal reducible matrix group a family of noncyclic matrices whose proportion is  $q^{-2} + O(q^{-3})$ .

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## 1 Introduction

This paper studies reducible matrix groups and matrix algebras over a finite field, say  $\mathbb{F}_q$  of order  $q$ , that is, some proper non-trivial subspace of the underlying vector space is fixed setwise by all the elements. If no such subspace exists the group or algebra is said to be *irreducible*. The first algorithm developed to test the irreducibility of a matrix algebra was the Parker MEATAXE based on the Norton Irreducibility Test described in [21]. Given generators for a matrix group or algebra the Parker Meataxe sought matrices of a special type by means of which it either proved irreducibility or constructed a non-trivial proper invariant subspace. The (non-zero) probability of failing to find suitable elements was not estimated.

The first analysis of a modified version of Parker's MEATAXE was due to Holt and Rees [12] in 1994, and an alternative modification of the irreducibility test using cyclic matrices was developed by Neumann and Praeger in [19]. Their Cyclic Irreducibility Test is a Las Vegas algorithm that verifies irreducibility of an irreducible matrix algebra by producing a cyclic matrix and a corresponding cyclic basis. Bounds on the error probability rely on estimates for the proportion of cyclic matrices in finite irreducible matrix algebras. Namely, in [17] they proved that the proportion of cyclic matrices in a full matrix algebra over  $\mathbb{F}_q$  is  $1 - q^{-3} + O(q^{-4})$  and obtained also, explicit lower bounds for this proportion in arbitrary irreducible matrix algebras, in [17, Theorem 5.5]. A motivation for the work presented in this paper was the problem of extending the scope of the

Cyclic MEATAXE to constructing an invariant subspace in the case of ‘large’ reducible matrix groups and algebras. Using the results of this paper such an extension of the Cyclic MEATAXE was developed and presented in [4, Chap 7].

A matrix is called *cyclic* if its characteristic polynomial equals its minimal polynomial. G.E. Wall [22] and Jason Fulman [6] independently calculated generating functions for the proportion of cyclic matrices in general linear groups and full matrix algebras, and determined exactly the limiting proportions as the dimension tends to infinity, namely for finite general linear groups  $\mathrm{GL}(n, q)$  (see [22, Equation 6.24] or [6, Theorem 8]) the limit as  $n$  tends to infinity is

$$\frac{1 - q^{-5}}{1 + q^{-3}} = 1 - q^{-3} + O(q^{-5}) \quad (1)$$

while for finite matrix algebras  $M(n, q)$  (see [22, Equation 6.23] and [6, Theorem 6]), the limit as  $n$  tends to infinity is

$$(1 - q^{-5}) \prod_{i=3}^{\infty} (1 - q^{-i}) = 1 - q^{-3} + O(q^{-4}). \quad (2)$$

The leading  $q$  term,  $q^{-3}$  in this case, is of significance as for large  $q$  this term dominates the later terms. Generalising these results, proportions of cyclic matrices have been computed for finite classical groups using both geometric methods [18] and generating function methods [2, 3, 6, 8].

Some matrix algebras contain no cyclic matrices, for example, the algebra of  $n \times n$  scalar matrices for any  $n \geq 2$ , while others contain a large proportion. One of the few results in the literature on proportions of cyclic matrices in reducible matrix groups and algebras is due to Jason Fulman [7]. Using cycle index methods he obtains the limiting proportion of cyclic matrices in the largest parabolic subgroup of a general linear group, namely the stabiliser of a 1-dimensional subspace. We reprove his result by a different method, and extend it for all subspace stabilisers in general linear groups and matrix algebras, yielding Theorem 1.1, the main result of this paper.

**Theorem 1.1.** *Let  $r, n \in \mathbb{Z}^+$  with  $r < n$ , let  $q$  be a prime power and let  $c_{\mathrm{GL},r}(n)$  and  $c_{\mathrm{M},r}(n)$  denote the proportions of cyclic matrices in the stabiliser in  $\mathrm{GL}(n, q)$  and  $M(n, q)$ , respectively, of an  $r$ -dimensional subspace of  $\mathbb{F}_q^n$ . Then*

$$c_{\mathrm{GL},r}(\infty) := \lim_{n \rightarrow \infty} c_{\mathrm{GL},r}(n) = 1 - q^{-2} + O(q^{-3})$$

and

$$c_{\mathrm{M},r}(\infty) := \lim_{n \rightarrow \infty} c_{\mathrm{M},r}(n) = 1 - q^{-2} + O(q^{-3}).$$

Moreover, for any  $d$  such that  $1 < d < q(q - 1)$ , we have that  $|c_{\mathrm{GL},r}(n) - c_{\mathrm{GL},r}(\infty)| = O(d^{-n})$  and  $|c_{\mathrm{M},r}(n) - c_{\mathrm{M},r}(\infty)| = O(d^{-n})$ .

The assertions about the limit and rate of convergence of the quantities  $c_{\mathrm{GL},r}(n)$  are proved in Theorems 4.14 and 4.22 in Section 4.2, while the assertions for  $c_{\mathrm{M},r}(n)$  are proved in Theorems 4.30 and 4.34 in Section 4.3.

Our approach is similar to that of Wall [22] in that we study the generating function for the proportions  $c_{\text{GL},r}(n)$  in order to calculate their limit as  $n$  approaches infinity. However, instead of producing an explicit expression for the generating function as an infinite product, we specify a finite series of steps involving partial differentiation and evaluation that produces the result. The number of steps is linear in  $r$ , and so for large (but fixed)  $r$  we do not write down an explicit form for the generating function. Nevertheless the information given is sufficient to determine the limiting proportion and rate of convergence as  $n$  tends to infinity. In Section 4.4 we demonstrate, for  $r = 1, 2$  how the procedure can be applied to give an explicit form for the generating function - the case of  $r = 1$  retrieving the result of Fulman [7].

Note that the exponent of the leading  $q$ -term  $-q^{-2}$  has increased by one over that for the general linear group case. Note also that this leading  $q$ -term is independent of the dimension  $r$  of the invariant subspace. Although this may suggest that the limiting proportion may be independent of the dimension of the invariant subspace, this is not the case. As we discuss in Remark 4.35, it is only the coefficient of  $q^{-2}$  that is independent of the dimension of the invariant subspace; the coefficient of  $q^{-3}$  changes for different values of  $r$ .

To better understand Theorem 1.1, we exhibit in Theorem 4.38, a family of noncyclic matrices in maximal reducible matrix groups with proportion  $q^{-2} + O(q^{-3})$ .

Results about the limits and rates of convergence of the proportion of cyclic matrices inside maximal completely reducible subgroups of  $\text{GL}(n, q)$  and maximal completely reducible subalgebras of  $\text{M}(n, q)$  are found in [4, Chapter 5], while similar results for separable matrices are found in [4, Chapter 6].

## 2 Preliminary Results

In this section we prove some preliminary results. Throughout this paper  $\mathbb{F}_q$  denotes a field of order  $q$ , and  $V = \mathbb{F}_q^n$  denotes the vector space of  $n$ -dimensional row vectors. The algebra of all  $n \times n$  matrices over  $\mathbb{F}_q$  is denoted  $\text{M}(n, q)$ , and the general linear group of all nonsingular  $n \times n$  matrices over  $\mathbb{F}_q$  by  $\text{GL}(n, q)$ . We sometimes write  $\text{M}(V) = \text{M}(n, q)$  or  $\text{GL}(V) = \text{GL}(n, q)$ . Each  $A \in \text{M}(n, q)$  acts naturally on  $V$  and for  $w \in V$ ,  $\langle w \rangle_A$  denotes the cyclic  $A$ -module generated by  $w$ , that is,  $\langle w \rangle_A = \langle w, wA, wA^2, \dots, wA^n \rangle$ . In particular if  $\langle w \rangle = V$  then  $w$  is called a *cyclic vector* for  $A$  and  $(w, A)$  is called a *cyclic pair* on  $V$ . It is well known that  $A$  is cyclic if and only if it has a cyclic vector. If  $W$  is an  $A$ -invariant subspace of  $V$  then  $A$  induces a matrix  $A|_W$  in  $\text{M}(W)$  and a matrix  $A|_{V/W}$  in  $\text{M}(V/W)$ . In particular the *nullspace*  $\text{null}(A)$  is an  $A$ -invariant subspace.

We denote the characteristic and minimal polynomial of  $A$  on  $V$  by  $c_A(t)$  and  $m_A(t)$  respectively.

## 2.1 Cyclic Matrices on Subspaces

**Lemma 2.1.** *Let  $X \in M(n, q)$ . Then  $X$ ,  $X^T$  and all conjugates of  $X$  have the same characteristic polynomial and the same minimal polynomial. In particular, they are either all non-cyclic or all cyclic.*

*Proof.* This is well known, see for example [5, Theorem 7.2 and Exercise 2 on p149] for the characteristic polynomial and [5, Exercise 4 on p150] for the minimal polynomial of similar, this is, conjugate matrices. The assertion for the transpose holds since for any polynomial  $f(t)$ , we have  $f(X)^T = f(X^T)$ .  $\square$

**Lemma 2.2.** *Let  $A$  be a cyclic matrix on  $V = \mathbb{F}_q^n$  with minimal polynomial  $m_A(t) = f(t)g(t)$  for monic polynomials  $f(t)$  and  $g(t)$ , and let  $w \in V$  be a cyclic vector for  $A$ . If  $W = \langle wf(A) \rangle_A$  then*

- (1) *the minimal polynomial of  $A|_W$  is  $g(t)$ ;*
- (2)  *$W$  has dimension  $n - \deg(f) = \deg(g)$ ;*
- (3)  *$W = \text{null}(g(A))$ .*

*Proof.* (1) Since  $(wf(A))g(A) = 0$ , the minimal polynomial  $h(t)$  of  $A|_W$  divides  $g(t)$ . Suppose that  $\deg(h) < \deg(g)$ . Then  $wf(A)A^i h(A) = 0$  for all  $i$ , and this implies that  $vf(A)h(A) = 0$  for all  $v \in \langle w \rangle_A = V$ . However  $f(t)h(t)$  is a polynomial of degree strictly less than  $\deg(m_A)$ , and we have a contradiction. Hence  $\deg(h) \geq \deg(g)$ , and since  $h(t)$  divides  $g(t)$  it follows that  $h(t) = g(t)$ .

(2) Since the minimal polynomial of  $A|_W$  has degree  $n - \deg(f)$  and the space  $W$  is cyclic as an  $A$ -module, it follows that  $W$  has dimension  $n - \deg(f)$ .

(3) Let  $u \in \langle wf(A) \rangle_A$ . There exists a positive integer  $k$  such that the vectors  $wf(A), wf(A)A, \dots, wf(A)A^k$  form a basis for  $\langle wf(A) \rangle_A$  and so  $u = \lambda_0 wf(A) + \lambda_1 wA f(A) + \dots + \lambda_k wA^k f(A)$  for some  $\lambda_0, \dots, \lambda_k \in \mathbb{F}_q$ . Then  $ug(A) = \lambda_0 wf(A)g(A) + \dots + \lambda_k wf(A)A^k g(A)$  and this equals 0 since every term is a multiple of  $wg(A)f(A) = wm_A(A) = 0$ . Hence  $u \in \text{null}(g(A))$  and  $\langle wf(A) \rangle_A \subseteq \text{null}(g(A))$ . In particular  $\dim(\text{null}(g(A))) \geq n - \deg(f) = \deg(g)$ .

Let us now look at the rank of  $g(A) = \sum_{i=0}^d a_i A^i$  where  $g(t) = \sum_{i=0}^d a_i t^i$  and  $d$  is the degree of  $g$ . Note in the following that  $a_d = 1$  since  $g(t)$  is monic.

$$\begin{aligned}
 wg(A) &= wA^d + \sum_{i=0}^{d-1} a_i wA^i \\
 wAg(A) &= wA^{d+1} + \sum_{i=0}^{d-1} a_i wA^{i+1} \\
 &\vdots \\
 wA^{n-d+1}g(A) &= wA^{n-1} + \sum_{i=0}^{d-1} a_i wA^{i+n-d-1}
 \end{aligned}$$

Now since  $wA^d, wA^{d+1}, \dots, wA^{n-1}$  are linearly independent, it follows from the equations above that  $wg(A), wAg(A), \dots, wA^{n-d-1}g(A)$  are linearly independent. So  $\text{rank}(g(A)) \geq n - d$  which means that  $\dim(\text{null}(g(A))) \leq d$ . By the previous paragraph equality holds and hence  $\langle wf(A) \rangle_A = \text{null}(g(A))$ .  $\square$

**Lemma 2.3.** *Let  $(w, A)$  be a cyclic pair on  $V = \mathbb{F}_q$ , let  $U$  be an  $A$ -invariant  $r$ -dimensional subspace of  $V$  and let  $c(t)$  be the characteristic polynomial of  $A|_{V/U}$ . Then  $c(t)$  divides  $c_A(t)$ ,*

- (1)  $U + w$  is a cyclic vector for  $A|_{V/U}$ ;
- (2)  $wc(A)$  is a cyclic vector for  $A|_U$ .

*Proof.* That  $c(t)$  divides  $c_A(t)$  is well known. By assumption  $\langle w \rangle_A = V$ , and hence  $\langle U + w \rangle_{A|_{V/U}} = V/U$ , proving (1).

By definition,  $c(A|_{V/U})$  is the zero matrix of  $V/U$ , that is to say  $c(t)$  is monic and  $c(A)|_{V/U} = 0$ . Thus  $wc(A) \in U$ . Since  $\deg(c) = \dim(V/U) = n - r$ , we have  $c(t) = t^{n-r} + \sum_{i=0}^{n-r-1} a_i t^i$  and  $wc(A) = wA^{n-r} + \sum_{i=0}^{n-r-1} a_i wA^i$ . Since  $(w, A)$  is a cyclic pair, the vectors  $wA, \dots, wA^{n-r}$  are linearly independent. Thus arguing as in the previous proof we see that the vectors  $wc(A), \dots, wc(A)A^{r-1}$  are also linearly independent, and each lies in  $U$  since  $U$  is  $A$ -invariant. They therefore form a basis for  $U$  since  $\dim(U) = r$ . Hence  $\langle wc(A) \rangle_A = U$  and (2) is proved.  $\square$

The next lemma describes all the invariant subspaces of a cyclic matrix whose characteristic polynomial has only one irreducible factor.

**Lemma 2.4.** *Let  $(w, A)$  be a cyclic pair on  $V = \mathbb{F}_q^n$ . Then the only  $A$ -invariant subspaces  $U$  of  $V$  are  $U = \langle wf(A) \rangle_A$  for some monic polynomial  $f(t)$  dividing  $m_A(t)$  and the minimal polynomial of  $A|_U$  on  $U$  is  $g(t)$  where  $m_A(t) = f(t)g(t)$ . Moreover, there is a one-to-one correspondence between  $A$ -invariant subspaces of  $V$  and monic divisors of  $m_A(t)$ .*

*Proof.* Let  $U$  be an  $A$ -invariant subspace of  $V$ . By Lemma 2.3,  $U + w$  is a cyclic vector for  $A|_{V/U}$ , and  $wf(A)$  is a cyclic vector for  $A|_U$ , where  $f(t) = c_{A|_{V/U}}(t) = m_{A|_{V/U}}(t)$  and  $f(t)$  divides  $m_A(t)$ , say  $m_A(t) = f(t)g(t)$ . Thus  $\langle wf(A) \rangle_A = U$  and  $\dim(U) = n - \deg(f) = \deg(g)$ . If also  $U = \langle wf'(A) \rangle_A$  where  $m_A(t) = f'(t)g'(t)$ , then by Lemma 2.2 (3),  $U = \text{null}(g(A)) = \text{null}(g'(A))$  and  $\deg(g) = \deg(g')$ . This implies that  $wf(A)g'(A) = 0$  whence  $m_A(t)$  divides  $f(t)g'(t)$ , or equivalently  $g(t)$  divides  $g'(t)$ . Thus  $g(t) = g'(t)$  and hence for distinct divisors  $f(t), f'(t)$  of  $m_A(t)$  the submodules  $\langle wf(A) \rangle_A$  and  $\langle wf'(A) \rangle_A$  are distinct. Also  $g(t) = m_{A|_U}(t)$ .  $\square$

## 2.2 Decomposition

For  $X \in M(n, q)$ , a *cyclic decomposition* of  $V = \mathbb{F}_q^n$  under  $X$  is a direct sum  $V = \bigoplus_{i=1}^{\ell} V_i$  where each  $V_i$  is  $X$ -cyclic and  $X$ -invariant, the minimal polynomial of  $X|_{V_i}$  divides the minimal polynomial of  $X|_{V_{i-1}}$  for  $1 < i \leq \ell$ , and the minimal polynomial of  $X|_{V_1}$  is the minimal polynomial of  $X$  (see [10, Lemma 11.7]). If  $X$  is cyclic, the cyclic decomposition of  $V$  under  $X$  is simply  $V = V_1$ .

Let  $A \in M(n, q)$  be cyclic with minimal polynomial  $m_A(t) = \prod_{i=1}^k f_i^{\alpha_i}(t)$  where the  $f_i$  are distinct monic irreducible polynomials and each  $\alpha_i > 0$ . The *primary decomposition* of  $V$  under  $A$  is  $V = \bigoplus_{i=1}^k V_i$  where for each  $i$  we have  $V_i = \langle w g_i(A) \rangle_A$  for  $g_i(t) = \frac{m(t)}{f_i^{\alpha_i}(t)}$ . With respect to an appropriate basis,  $A$  can be decomposed into blocks or components as below:

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & A_k \end{pmatrix} \quad \text{which we write as} \quad A_1 \oplus \cdots \oplus A_k$$

where each  $A_i$  is cyclic on  $V_i$  with minimal polynomial  $f_i^{\alpha_i}$  (see [10, Lemma 11.8]).

The proof of the following lemma is straight forward and is omitted.

**Lemma 2.5.** *Let  $A \in M(n, q)$  be a cyclic matrix on  $V = \mathbb{F}_q^n$  and let  $V = \bigoplus V_i$  be its cyclic (respectively primary) decomposition, and let  $B \in \text{GL}(V)$ . Then*

(a) *The conjugate  $B^{-1}AB$  has cyclic (respectively primary) decomposition  $V = \bigoplus (V_i B)$ .*

(b) *If  $U_i$  is an  $A$ -invariant subspace of  $V_i$  then  $U_i B$  is a  $B^{-1}AB$ -invariant subspace of  $V_i B$  of the same dimension and  $m_{A|_{U_i}}(t)$  is the minimal polynomial of  $(B^{-1}AB)|_{U_i B}$ .*

(c) *Moreover,  $A$  and  $A' \in M(n, q)$  are conjugate under an element of  $\text{GL}(n, q)$  if and only if the sequences of minimal polynomials induced on the spaces in their cyclic decompositions are the same.*

Note that if  $\bigoplus_{i=1}^s V_i$  is the cyclic decomposition for  $A \in M(n, q)$  and  $m_{A|_{V_i}}(t) = \sum_{j=0}^{d_i} a_{ij} t^j$  with  $a_{id_i} = 1$ , then  $A$  is conjugate to  $A_1 \oplus \cdots \oplus A_s$  where for each  $i$ ,  $A_i$  is conjugate under  $\text{GL}(V_i)$  to the companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{i0} & -a_{i1} & -a_{i2} & \cdots & -a_{i(d_i-1)} \end{pmatrix}$$

**Corollary 2.6.** *Cyclic matrices  $X$  and  $Y$  in  $M(n, q)$  have the same minimal polynomial if and only if they are conjugate in  $\text{GL}(n, q)$ .*

This corollary follows from the fact that the cyclic decomposition of a vector space under a cyclic matrix has just one component. It implies that there is exactly one  $\text{GL}(n, q)$ -conjugacy class of cyclic matrices with a given characteristic polynomial.

**Lemma 2.7.** *Let  $A \in M(n, q)$  acting on  $V = \mathbb{F}_q^n$  and suppose that  $V = \bigoplus_{i \leq k} V_i$  where each  $V_i$  is  $A$ -invariant. Let  $c_i(t)$  and  $m_i(t)$  be the characteristic and minimal polynomials respectively of  $A$  on  $V_i$  for  $i \leq k$ . Then  $A$  is cyclic if and only if each  $A|_{V_i}$  is cyclic and  $\gcd(c_i, c_{i'}) = 1$  for all  $i \neq i'$ .*

*Proof.* Suppose that each  $A|_{V_i}$  is cyclic and that  $\gcd(c_i, c_{i'}) = 1$  for all  $i \neq i'$ . Then  $m_i = c_i$  for all  $i$  so  $\text{lcm}(m_1, \dots, m_k) = \text{lcm}(c_1, \dots, c_k) = f$ , say. Since  $\gcd(c_i, c_{i'}) = 1$  for all  $i \neq i'$  we have that  $f = c_1 \dots c_k = c$  and we know that  $\text{lcm}(m_1, \dots, m_k) = m$ . Hence  $c = m$  and  $A$  is cyclic.

Conversely suppose that  $A$  is cyclic. By Lemma 2.3, each of the  $A|_{V_i}$  is cyclic. Suppose that  $\gcd(c_i, c_{i'}) \neq 1$  for some  $i \neq i'$ . Then  $f(t) := \text{lcm}(c_1, \dots, c_k)$  divides  $\frac{c_1 \dots c_k}{\gcd(c_i, c_{i'})}$  so  $\deg(f) < n$  and each  $m_i$  divides  $c_i$  which, in turn, divides  $f$ . Each vector  $v \in V$  can be written as  $v = v_1 + \dots + v_k$  for some  $v_i \in V_i$  for  $i \leq k$ . Then  $vf(A) = v_1f(A) + \dots + v_kf(A) = 0$  since each  $m_i$  divides  $f$ . Hence  $m$  divides  $f$  which has degree less than  $n$  so  $m \neq c$  which is a contradiction with  $A$  being cyclic on  $V$ . Hence  $\gcd(c_i, c_{i'}) = 1$  for all  $i \neq i'$ .  $\square$

**Lemma 2.8.** *Let  $A$  be a cyclic matrix on  $V$  with minimal polynomial  $\prod f_i^{\alpha_i}$  and let  $V = \oplus V_i$  be the primary decomposition of  $V$  under  $A$ . Let  $U$  be an  $A$ -invariant subspace of  $V$ . Then the primary decomposition for  $U$  under  $A|_U$  is  $\oplus U_i$ , where for each  $i$ ,  $U_i = U \cap V_i$ .*

*Proof.* Let  $v$  be a cyclic vector for  $A$  in  $V$ . Using the notation introduced at the beginning of Section 2.2, let  $c_A(t) = f_i^{\alpha_i}(t)g_i(t)$  for each  $i$ , so that by Lemma 2.4,  $V_i = \langle v_i \rangle_A$  where  $v_i = vg_i(A)$ . Let  $c_{A|_U}(t) = \prod f_i^{\beta_i}(t)$ , and let  $c_A(t) = c_{A|_U}(t)g(t)$  and  $c_{A|_U}(t) = f_i^{\beta_i}(t)g'_i(t)$  for each  $i$ , where  $f_i$  does not divide  $g'_i$ . Note that  $g_i(t)$  divides  $g'_i(t)g(t)$  for each  $i$ .

Again by Lemma 2.4,  $U = \langle u \rangle_A$  where  $u = vg(A)$  and  $U_i := \langle ug'_i(A) \rangle_A$  is  $A$ -invariant with  $c_{A|_{U_i}}(t) = f_i^{\beta_i}(t)$ . Thus  $U = \oplus_{i=1}^s U_i$  is the primary decomposition of  $U$  under  $A|_U$ , where  $U_i$  may be 0 for some  $i$ . Also, since  $g_i$  divides  $g'_i g$  it follows that  $ug'_i(A) = vg(A)g'_i(A) \in \langle vg_i(A) \rangle_A = V_i$ , so  $U_i \subseteq V_i \cap U$ . Then  $\oplus_{j \neq i} U_j \subseteq \oplus_{j \neq i} V_j$  and so is disjoint from  $V_i \cap U$ . This implies that  $U_i = V_i \cap U$  for each  $i$ .  $\square$

We are working with matrices on  $V$  which fix a proper subspace of  $V$  so we need convenient notation for the group and the algebra of such matrices. For  $V = \mathbb{F}_q^n$ , denote  $\text{GL}(V) := \text{GL}(n, q)$  and  $\text{M}(V) := \text{M}(n, q)$ . For a subspace  $U \leq V$ , let  $\text{GL}(V)_U$  be the subgroup of  $\text{GL}(V)$  which consists of all matrices that fix  $U$  setwise. Similarly let  $\text{M}(V)_U$  be the set of all matrices in  $\text{M}(V)$  which leave the subspace  $U$  invariant.

**Lemma 2.9.** *Let  $A, A'$  be cyclic matrices in  $\text{M}(V)_U$ . Then  $A$  and  $A'$  are conjugate by an element of  $\text{GL}(V)_U$  if and only if both  $m_A = m_{A'}$  and  $m_{A|_U} = m_{A'|_U}$ .*

*Proof.* If  $A$  and  $A'$  are  $\text{GL}(V)_U$ -conjugate then also  $A|_U$  and  $A'|_U$  are  $\text{GL}(U)$ -conjugate, and hence by Lemma 2.1,  $c_A = c_{A'}$  and  $c_{A|_U} = c_{A'|_U}$ . By Lemma 2.3,  $A|_U$  and  $A'|_U$  are cyclic also, and hence  $m_A = c_A = c_{A'} = m_{A'}$  and similarly  $m_{A|_U} = m_{A'|_U}$ .

Conversely suppose that  $m_A(t) = m_{A'}(t)$  and  $m_{A|_U}(t) = m_{A'|_U}(t)$ . Then by Corollary 2.6, there exists  $X \in \text{GL}(V)$  such that  $X^{-1}AX = A'$ . By assumption  $U$  is an  $A'$ -invariant subspace and the minimal polynomial of  $A'|_U$  is  $m_{A'|_U} =$



$m_{A|_U}$ . We also have that  $UX$  is  $A'$ -invariant (since  $A' = X^{-1}AX$ ) and the minimal polynomial of  $A'$  on  $UX$  is  $m_{A|_U}$ . By Lemma 2.4, it follows that  $U = UX$  since the minimal polynomial of  $A'$  on each space is the same. Hence  $X \in \text{GL}(V)_U$ .  $\square$

This lemma characterises the set of conjugacy classes of cyclic matrices in  $M(V)_U$  under the action of  $\text{GL}(V)_U$ . There is one conjugacy class for each pair  $f, h$  where  $h$  is a monic degree  $n$  polynomial,  $f$  is a monic degree  $r$  polynomial dividing  $h$  and  $r = \dim(U)$ . For matrices in  $\text{GL}(V)_U$ , we in addition require  $f$  and  $h$  to have a nonzero constant term.

### 2.3 Centralisers

This section describes the centralisers of cyclic matrices in  $\text{GL}(V)_U$ . For  $A \in M(V)$ , let  $C_{\text{GL}(V)}(A)$  denote the centraliser of  $A$  in  $\text{GL}(V)$ .

**Lemma 2.10.** *Let  $V = \mathbb{F}_q^n$ ,  $U$  be a subspace of  $V$ , and  $A \in M(V)_U$  be cyclic. Then  $C_{\text{GL}(V)}(A) \leq \text{GL}(V)_U$ .*

*Proof.* Let  $B \in C_{\text{GL}(V)}(A)$ , so  $B^{-1}AB = A$ . Let  $w$  be a cyclic vector for  $A$ . By Lemma 2.4,  $U = \langle wf(A) \rangle_A$  for some monic  $f(t)$  dividing  $m_A(t)$  and the minimal polynomial of  $A$  on  $U$  is  $g(t)$  where  $m_A(t) = f(t)g(t)$ . The subspace  $UB$  is also invariant under  $B^{-1}AB = A$  and, since  $Ug(A) = 0$ ,  $UB$  is annihilated by  $B^{-1}g(A)B = g(B^{-1}AB) = g(A)$ . Hence  $UB \subseteq \text{null}(g(A))$  and by Lemma 2.2(3),  $\text{null}(g(A)) = \langle wf(A) \rangle_A = U$ . Then since  $\dim(UB) = \dim(U)$  we have that  $UB = U$ . Hence  $B$  fixes  $U$  and  $B \in \text{GL}(V)_U$ .  $\square$

A proof of the next lemma may be found in [17, Corollary 2.3 and Remark 2.6].

**Lemma 2.11.** *Let  $n = ij$  and suppose that  $p$  is an irreducible polynomial of degree  $i$  over  $\mathbb{F}_q$ , and  $A \in M(n, q)$  is a cyclic matrix with characteristic polynomial  $p^j$ . Then  $|C_{\text{GL}(n, q)}(A)| = q^{ij}(1 - q^{-i})$ .*

The statement of Lemma 2.11 is for an arbitrary irreducible degree  $i$  polynomial. Therefore the size of the centraliser of any matrix with characteristic polynomial the  $j$ th power of irreducible degree  $i$  polynomial, depends only on  $i$  and  $j$ , and we set

$$\text{Cent}(i, j) := q^{ij}(1 - q^{-i}) \quad (3)$$

so that, by Lemma 2.11,  $\text{Cent}(i, j) = |C_{\text{GL}(V)}(A)|$  for any cyclic matrix  $A$  with characteristic polynomial  $p^j$  for some monic degree  $i$  irreducible polynomial  $p$ . We will often write  $\text{Cent}(p^j)$  in this instance also.

**Lemma 2.12.** *Let  $f$  and  $g$  be distinct monic irreducible polynomials and let  $a, b \in \mathbb{Z}^+$ . Then  $\text{Cent}(f^a)\text{Cent}(g^b) = \text{Cent}(f^a g^b)$ .*

*Proof.* Let  $A$  be a cyclic matrix over a vector space  $V$  with minimal polynomial  $f^a g^b$ . Then the primary decomposition of  $V$  under  $A$  is  $V = V_1 \oplus V_2$  where  $A|_{V_1}$  is cyclic with minimal polynomial  $f^a$  and  $A|_{V_2}$  is cyclic with minimal polynomial  $g^b$ .

By Lemma 2.10,  $C_{\text{GL}(V)}(A) \subseteq \text{GL}(V)_{V_1} \cap \text{GL}(V)_{V_2} \cong \text{GL}(V_1) \times \text{GL}(V_2)$  and hence  $C_{\text{GL}(V)}(A) \cong C_{\text{GL}(V_1)}(A|_{V_1}) \times C_{\text{GL}(V_2)}(A|_{V_2})$ . It follows that  $\text{Cent}(f^a g^b) = \text{Cent}(f^a) \text{Cent}(g^b)$ .  $\square$

An analogous proof gives the following general result.

**Corollary 2.13.** *Let  $f = \prod_{i=1}^k f_i^{\alpha_i}$  where the  $f_i$  are pairwise distinct monic irreducible polynomials of degree  $d_i$ . Then*

$$\text{Cent}(f) = \prod_{i=1}^k \text{Cent}(f_i^{\alpha_i}) = \prod_{i=1}^k \text{Cent}(d_i, \alpha_i).$$

### 3 The work of Wall

In this section we outline G.E. Wall's [22] generating function method for determining the limiting proportion of cyclic matrices in  $\text{GL}(V)$  and  $\text{M}(V)$ . Our notation is based on that used in [22].

#### 3.1 Cyclic Matrices in General Linear Groups

Let

$$C_{\text{GL}}(t) = 1 + \sum_{n=1}^{\infty} c_{\text{GL}}(n) t^n$$

be the generating function for the proportion of cyclic matrices in  $\text{GL}(V)$  where for each  $n$ ,  $V = \mathbb{F}_q^n$  and  $c_{\text{GL}}(n)$  denotes the proportion of cyclic matrices in  $\text{GL}(V)$ .

Let  $\Gamma_{\text{GL}}(n)$  be the set of all cyclic matrices in  $\text{GL}(V)$ . Then by Lemma 2.1 there is a one-to-one correspondence between the set of orbits of  $\text{GL}(V)$  in its action on  $\Gamma_{\text{GL}}(n)$  by conjugation and the set of monic degree  $n$  polynomials  $h$  over  $\mathbb{F}_q$  such that  $h(0) \neq 0$ . Denote the orbit on  $\Gamma_{\text{GL}}(n)$  consisting of all matrices with minimal polynomial  $h$  by  $\Gamma_h$ . By the Orbit-Stabiliser Theorem

$$|\Gamma_h| = \frac{|\text{GL}(V)|}{|\text{Cent}(h)|}$$

where  $\text{Cent}(h) = |C_{\text{GL}(V)}(A)|$  for a matrix  $A \in \Gamma_h$ . To enable us to focus on irreducible polynomials of a given degree, we make the following definition.

**Definition 3.1.** Let  $\mathcal{P}^+$  be the set of all monic polynomials over  $\mathbb{F}_q$  with a nonzero constant term (including 1) and let  $\mathcal{P}_i^+$  be the subset of  $\mathcal{P}^+$  containing the constant polynomial 1 and those polynomials whose irreducible factors all have degree  $i$ .

In light of this definition it follows that

$$c_{\text{GL}}(n) = \sum_{\substack{h \in \mathcal{P}^+ \\ \deg(h)=n}} \frac{|\Gamma_h|}{|\text{GL}(V)|} = \sum_{\substack{h \in \mathcal{P}^+ \\ \deg(h)=n}} \frac{1}{\text{Cent}(h)}. \quad (4)$$

For each monic irreducible polynomial  $p$  of degree  $i$ , define the formal power series

$$F^p = F^p(t) := 1 + \sum_{j=1}^{\infty} \frac{t^{ij}}{\text{Cent}(p^j)}. \quad (5)$$

Using (4), the product of the  $F^p$  over all monic irreducible polynomials  $p$  excluding  $t$  is

$$\prod_{p \neq t} F^p = \sum_{h \in \mathcal{P}^+} \frac{t^{\deg(h)}}{\text{Cent}(h)} = 1 + \sum_{n=1}^{\infty} c_{\text{GL}}(n) t^n = C_{\text{GL}}(t). \quad (6)$$

**Definition 3.2.** Let  $N(i, q)$  be the number of monic degree  $i$  irreducible polynomials over  $\mathbb{F}_q$  and let  $N^+(i, q)$  be the number of polynomials in  $N(i, q)$  with nonzero constant term.

Thus  $N^+(i, q) = N(i, q)$  if  $i = 1$  and  $N(1, q) = N^+(1, q) + 1 = q$ . Note that  $F^p(t)$  depends only on the degree and multiplicity of the irreducible factors of  $p$ . Let  $F_i(t)$  denote the following formal power series.

$$F_i = F_i(t) := \prod_{\substack{\deg(p)=i \\ p \neq t}} F^p(t) = \left( 1 + \sum_{j=1}^{\infty} \frac{t^{ij}}{\text{Cent}(i, j)} \right)^{N^+(i, q)}. \quad (7)$$

It follows that

$$C_{\text{GL}}(t) = \prod_{p \neq t} F^p = \prod_{i=1}^{\infty} F_i. \quad (8)$$

The following lemma underpins the proof of Wall's convergence result in Theorem 3.4 and we will also use it in Section 4.

**Lemma 3.3.** Suppose that  $g(u) = \sum_{n \geq 0} a_n u^n$  and  $g(u) = f(u)/(1-u)$  for  $|u| < 1$ . If  $f(u)$  is analytic in a disc of radius  $R$ , where  $R > 1$ , then  $\lim_{n \rightarrow \infty} a_n = f(1)$  and  $|a_n - f(1)| = O(d^{-n})$  for any  $d$  such that  $1 < d < R$ .

Wall proved that  $C_{\text{GL}}(t)$  is convergent for  $|t| < 1$  and that  $(1-t)C_{\text{GL}}(t)$  is convergent for  $|t| < q^2$ . Hence by Lemma 3.3 the limit of the coefficients of  $C_{\text{GL}}(t)$  is equal to  $(1-t)C_{\text{GL}}(t)$  evaluated at  $t = 1$ . This, and the rate of convergence, was calculated by Wall and is given in Theorem 3.4.

**Theorem 3.4.**  $\lim_{n \rightarrow \infty} c_{\text{GL}}(n) = c_{\text{GL}}(\infty)$  exists and satisfies

$$c_{\text{GL}}(\infty) = \frac{1 - q^{-5}}{1 + q^{-3}} = 1 - q^{-3} + O(q^{-5})$$

and  $|c_{\text{GL}}(n) - c_{\text{GL}}(\infty)| = O(d^{-n})$  for any  $d$  with  $1 < d < q^{-2}$ .

### 3.2 Cyclic Matrices in Full Matrix Algebras

Let  $\Gamma_{\text{M}}(n)$  be the set of all cyclic matrices in  $\text{M}(V)$ , where  $V = \mathbb{F}_q^n$ . Then by Lemma 2.1 there is a one-to-one correspondence between the set of orbits of  $\text{GL}(V)$  in its action on  $\Gamma_{\text{M}}(n)$  by conjugation and the set of monic degree  $n$  polynomials over  $\mathbb{F}_q$ . Denote the orbit on  $\Gamma_{\text{M}}(n)$  consisting of all matrices with minimal polynomial  $h$  by  $\Gamma_h$ . Note that  $\Gamma_{\text{GL}}(n) \subset \Gamma_{\text{M}}(n)$  and that when  $h$  has nonzero constant term, the orbit  $\Gamma_h$  defined in Section 3.1 is the same as the orbit  $\Gamma_h$  we use here. By the Orbit-Stabiliser Theorem

$$|\Gamma_h| = \frac{|\text{GL}(V)|}{\text{Cent}(h)}$$

**Definition 3.5.** Let  $\mathcal{P}$  be the set of all monic polynomials over  $\mathbb{F}_q$  (including 1) and let  $\mathcal{P}_i$  be the subset of  $\mathcal{P}$  consisting of the constant polynomial 1 and all polynomials whose irreducible factors all have degree  $i$ .

Recall that  $\mathcal{P}_i^+$  consists of 1 and all monic polynomials with nonzero constant term whose irreducible factors all have degree  $i$ . Note that  $\mathcal{P}_i^+ = \mathcal{P}_i$  for  $i \geq 2$  but we use the notation for convenience.

**Definition 3.6.** Let  $\omega(n) = \frac{|\text{GL}(n, q)|}{|\text{M}(n, q)|}$ .

It is easy to show that

$$\omega(n) = \prod_{i=1}^n (1 - q^{-i}). \quad (9)$$

Denote by  $c_{\text{M}}(n)$  the proportion of cyclic matrices in  $\text{M}(V)$ . It follows that

$$c_{\text{M}}(n) = \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{|\Gamma_h|}{|\text{M}(V)|} = \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{|\Gamma_h| |\text{GL}(V)|}{|\text{M}(V)| |\text{GL}(V)|} = \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{\omega(n)}{\text{Cent}(h)} \quad (10)$$

and hence

$$\frac{c_M(n)}{\omega(n)} = \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{1}{Cent(h)}$$

which can be calculated by a similar method to that used for calculating  $c_{GL}(n)$ .

The following is a ‘weighted’ generating function for the proportion of cyclic matrices in full matrix algebras over  $\mathbb{F}_q$ , and is considered by Wall:

$$C_M(t) = 1 + \sum_{n=1}^{\infty} \left( \frac{c_M(n)}{\omega(n)} \right) t^n. \quad (11)$$

The product of the functions  $F_i$  defined in (7) gave

$$\prod_i F_i(t) = \sum_{h \in \mathcal{P}^+} \frac{t^{\deg(h)}}{Cent(h)}$$

but in Section 3.1,  $F_1(t)$  did not include the power series for the polynomial  $t$ . Including the polynomial  $t$  in Equation (6) gives

$$F^t(t) \prod_i F_i(t) = \sum_{h \in \mathcal{P}} \frac{t^{\deg(h)}}{Cent(h)} = 1 + \sum_{i=1}^{\infty} \left( \frac{c_M(n)}{\omega(n)} \right) t^n = C_M(t).$$

It follows that

$$\begin{aligned} C_M(t) &= \sum_{h \in \mathcal{P}} \frac{t^{\deg(h)}}{Cent(h)} \\ &= F^t(t) \prod_i F_i(t) \\ &= \left( 1 + \sum_{j=1}^{\infty} \frac{t^j}{Cent(t^j)} \right) C_{GL}(t) \quad (\text{by (8)}) \\ &= \left( 1 + \sum_{j=1}^{\infty} \frac{t^j}{q^j(1-q^{-1})} \right) C_{GL}(t) \\ &= \left( 1 + \frac{t}{(q-1)(1-tq^{-1})} \right) C_{GL}(t). \end{aligned}$$

To calculate the limiting proportion of the coefficients of  $C_M(t)$  as  $n$  tends to infinity, we use Lemma 3.3 to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{c_M(n)}{\omega(n)} &= (C_M(t)(1-t))|_{t=1} \\ &= \left( C_{GL}(t)(1-t) \left( 1 + \frac{t}{(q-1)(1-tq^{-1})} \right) \right) |_{t=1} \\ &= \frac{1-q^{-5}}{1+q^{-3}} \times \frac{1+q^{-3}}{(1-q^{-1})(1-q^{-2})} \\ &= \frac{1-q^{-5}}{(1-q^{-1})(1-q^{-2})} \end{aligned}$$

where Theorem 3.4 and [4, Lemma 2.5.2] are used to justify the second last line. Since  $\lim_{n \rightarrow \infty} \omega(n)$  exists, the limit of  $c_M(n)$  as  $n$  tends to infinity is

$$\begin{aligned}
\lim_{n \rightarrow \infty} c_M(n) &= \lim_{n \rightarrow \infty} \omega(n) \times \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \\
&= \prod_{i=1}^{\infty} (1 - q^{-i}) \times \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \\
&= (1 - q^{-5}) \prod_{i=3}^{\infty} (1 - q^{-i}).
\end{aligned}$$

Hence we have the following theorem, proved by Wall [22, Equation 6.23]. Wall also gave information about the rate of convergence of the  $c_M(n)$  to  $c_M(\infty)$ .

**Theorem 3.7.**  $\lim_{n \rightarrow \infty} c_M(n) = c_M(\infty)$  exists and satisfies

$$c_M(\infty) = (1 - q^{-5}) \prod_{i=3}^{\infty} (1 - q^{-i}) = 1 - q^{-3} + O(q^{-4})$$

and  $|c_M(n) - c_M(\infty)| = O(d^{-n})$  for any  $d$  with  $1 < d < q^{-2}$ .

## 4 Maximal Reducible Groups and Algebras

This section is devoted to cyclic matrices in maximal reducible matrix groups. We calculate the generating function for the proportions of cyclic matrices inside such groups and we calculate the limiting proportion of cyclic matrices as the size of the matrices tends to infinity. Then we modify our procedures to calculate the generating function and limiting proportion of cyclic matrices in the corresponding maximal reducible matrix algebras.

### 4.1 The Generating Function

Let  $V = \mathbb{F}_q$  and let  $U$  be an  $r$ -dimensional subspace of  $V$ . We investigate cyclic matrices in the setwise stabiliser,  $\text{GL}(V)_U$ , of  $U$  in  $\text{GL}(V)$ , using similar notation to (4). Let  $\Gamma_{\text{GL},r}(n)$  be the set of all cyclic elements of  $\text{GL}(V)_U$ . Then  $\Gamma_{\text{GL},r}(n)$  is invariant under conjugation by elements of  $\text{GL}(V)_U$  and by Lemma 2.9, there is a one-to-one correspondence between the set of orbits on  $\Gamma_{\text{GL},r}(n)$  under this conjugation action of  $\text{GL}(V)_U$  and the set of pairs of monic polynomials  $(f, h)$  over  $\mathbb{F}_q$  where  $\deg(f) = r$ ,  $\deg(h) = n$ ,  $f$  divides  $h$  and  $h(0) \neq 0$ . Hence all cyclic matrices  $A$  in  $\text{GL}(V)_U$ , such that  $m_{A|_U} = f$  and  $m_{A|_V} = h$ , lie in the same  $\text{GL}(V)_U$ -orbit which we denote by  $\Gamma_{f,h}$ . Also, by Lemma 2.10 the centraliser in  $\text{GL}(V)_U$  of a cyclic matrix  $A$  in  $\Gamma_{f,h}$  is equal to the centraliser in  $\text{GL}(V)$  of  $A$ , and hence has order  $\text{Cent}(h)$  (as defined in Section 2.3). We have that

$$|\Gamma_{f,h}| = \frac{|\text{GL}(V)_U|}{\text{Cent}(h)} \quad (12)$$

and hence  $|\Gamma_{f,h}|$  is the same for any  $f$  of degree  $r$  dividing  $h$ .

Since any  $r$ -dimensional subspace of  $V$  can be mapped to any other  $r$ -dimensional subspace of  $V$  by an element of  $\text{GL}(V)$ , the size  $|\Gamma_{f,h}|$  is independent

of the subspace  $U$  of  $V$ , depending only on the dimension  $r$ . So without any loss of generality let  $c_{\text{GL},r}(n)$  be the proportion of cyclic matrices in  $\text{GL}(V)_U$  and let

$$C_{\text{GL},r}(t) = \sum_{n=r}^{\infty} c_{\text{GL},r}(n)t^n$$

be the associated generating function. Note that the terms start from  $n = r$  because  $c_{\text{GL},r}(n)$  is not defined for  $n < r$ .

**Definition 4.1.** Let  $\alpha(h; r)$  denote the number of distinct monic degree  $r$  factors of  $h$ .

Note that  $\alpha(h; r)$  will be zero if the degree of  $h$  is less than  $r$ . Recall that  $\mathcal{P}^+$  is the set of monic polynomials over  $\mathbb{F}_q$  with nonzero constant term. Since  $|\Gamma_{f,h}|$  is the same for any degree  $r$  factor of  $h$  we can write

$$c_{\text{GL},r}(n) = \sum_{\substack{h \in \mathcal{P}^+ \\ \deg(h)=n \\ \deg(f)=r \\ f|h}} \frac{|\Gamma_{f,h}|}{|\text{GL}(V)_U|} = \sum_{\substack{h \in \mathcal{P}^+ \\ \deg(h)=n}} \frac{\alpha(h; r)|\Gamma_{f,h}|}{|\text{GL}(V)_U|} = \sum_{\substack{h \in \mathcal{P}^+ \\ \deg(h)=n}} \frac{\alpha(h; r)}{\text{Cent}(h)} \quad (13)$$

where in the summation  $f$  is a monic degree  $r$  divisor of  $h$ . If there were an easy way to calculate  $\alpha(h; r)$  then calculating  $c_{\text{GL},r}(n)$  would be simple but unfortunately there is not.

For each monic irreducible polynomial  $p$  of degree  $i$  we will again form the power series  $F^p$  as defined in Equation (5) and let  $F_i$  be as defined in Equation (7). We saw in Section 3.1 that

$$\prod_{i=1}^{\infty} F_i = \sum_{h \in \mathcal{P}^+} \frac{t^{\deg(h)}}{\text{Cent}(h)}.$$

This is almost the generating function we require except that we need a factor  $\alpha(h; r)$  in the term corresponding to  $h$ , for each  $h$ .

We provide two lemmas before the methodology for calculating a new expression for  $C_{\text{GL},r}(t)$ .

**Lemma 4.2.** For  $b \in \mathbb{Z}^+$ ,  $\sum_{m \geq 0} \frac{m^b t^{mk}}{k^m m!} = \frac{t}{k} \frac{d}{dt} \left( \dots \left( \frac{t}{k} \frac{d}{dt} \left( e^{t^k/k} \right) \right) \right)$ , where the differentiation is performed  $b$  times.

*Proof.* Since  $e^{t^k/k} = \sum_{m \geq 0} \frac{t^{km}}{k^m m!}$ , we have  $\frac{t}{k} \frac{d}{dt} (e^{t^k/k}) = \sum_{m \geq 0} \frac{m t^{km}}{k^m m!}$  and repeated differentiation gives the result.  $\square$

**Lemma 4.3.**

$$\frac{1}{m!} \frac{d^m}{dx^m} \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \binom{n}{m} a_n x^{n-m}.$$

*Proof.* Differentiating the power series gives  $\sum_{n \geq 0} a_n n(n-1) \dots (n-m+1) x^{n-m}$ . Now  $\frac{n(n-1) \dots (n-m+1)}{m!} = \binom{n}{m}$  so on noting that  $\binom{n}{m} = 0$  if  $n < m$ , we get the result.  $\square$

For a given  $r$ , the partitions of  $r$  correspond to all the ways we can obtain a monic degree  $r$  polynomial as a product of irreducible polynomials. For example, the partitions of 2 are  $\{2\}$  and  $\{1, 1\}$ . These correspond to a monic irreducible degree 2 polynomial and a product of two monic irreducible degree 1 polynomials respectively. The problem arises when we have two irreducible polynomials of the same degree dividing our degree  $r$  polynomial, as is possible in the  $\{1, 1\}$  case. We can't just choose two polynomials out of all the degree 1 factors because then we'll be over-counting if we have polynomials with multiplicity greater than one. So we need to break this case down into two sub-cases. In one sub-case we choose two different degree 1 polynomials and in the other sub-case we choose one degree 1 polynomial of multiplicity at least 2. These choices again correspond to the partitions of 2 but this time the partitions indicate the multiplicity of our factor not the degree.

Rather than taking partitions of the components of the original partition, we create a two-dimensional array whilst keeping in mind what we are counting. We denote such an array by  $(m_{ij})$ , where  $m_{ij}$  denotes the number of monic irreducible degree  $i$  factors of  $h$  that divide our degree  $r$  polynomial  $f$  with multiplicity  $j$ . The constraints on  $(m_{ij})$  are that we need the total degree of all the polynomials dividing  $f$  to equal  $r$  and we need, for each  $i$ , the total number of distinct degree  $i$  irreducible polynomials dividing  $f$  to be no more than  $N(i, q)$  (see Definition 3.2). We formulate these constraints as follows.

**Definition 4.4.** Let  $\mathcal{M}(r)$  be the set of all two-dimensional integer arrays,  $M = (m_{ij})$ , such that

1.  $\sum_{i,j} ij m_{ij} = r$ ;
2.  $\sum_j m_{ij} \leq N(i, q)$  for all  $i$ ; and
3.  $m_{ij} \geq 0$  for all  $i, j$ .

Let  $\mathcal{M}_{part}(r)$  be the subset of  $\mathcal{M}(r)$  consisting of all  $(m_{ij})$  for which  $m_{ij} = 0$  whenever  $j \geq 2$ .

Note that, by parts (1) and (3),  $m_{ij} = 0$  whenever  $ij > r$  so these arrays are finite and could have been defined as  $r \times r$  arrays.

We will often work with just those arrays  $(m_{ij})$  for which  $m_{ij} = 0$  whenever  $j \geq 2$  so that  $(m_{ij})$  is essentially a "column vector". These correspond to polynomials  $f$  having no irreducible factors with multiplicity more than 1, that is to say, *separable polynomials*. Moreover such an  $(m_{ij})$  corresponds to a partition of  $r$  with  $m_{i1}$  parts of size  $i$ , for each  $i$ . Hence  $\mathcal{M}_{part}(r)$  contains those  $(m_{ij})$  which correspond to partitions of  $r$ .

Each degree  $r$  factor  $f$  of  $h$  corresponds to a unique  $M = M(f) = (m_{ij}) \in \mathcal{M}(r)$  defined as follows. Let  $m_{ij}$  be the number of distinct monic irreducible



degree  $i$  factors of  $f$  of multiplicity  $j$ . Denote by  $\alpha(h; r, M)$  the number of degree  $r$  factors  $f$  of  $h$  such that  $M(f) = M$ . If  $M(f) = M$  we will say that  $f$  corresponds to  $M$ . We can break down the problem of computing  $\alpha(h; r)$  into computing  $\alpha(h; r, M)$  for each  $M \in \mathcal{M}(r)$  and summing over  $M$ , that is,

$$\alpha(h; r) = \sum_{M \in \mathcal{M}(r)} \alpha(h; r, M).$$

Now we concentrate on a fixed  $M \in \mathcal{M}(r)$ . We have to make sure that, when selecting irreducible polynomials as factors of  $f$ , we do not choose one we have already chosen to be in  $f$  or else that polynomial will have the incorrect multiplicity. The choices of irreducible degree  $i$  polynomials of various multiplicities are independent for each  $i$  so we break down the problem even further.

Let  $h \in \mathcal{P}^+$  and for each  $i$ , let  $h_i$  be the product of all the monic irreducible degree  $i$  factors of  $h$ , including multiplicities. If  $h$  has no degree  $i$  factors for some  $i$ , then set  $h_i := 1$ . Thus  $h_i \in \mathcal{P}_i^+$  for all  $i$ , and  $h = h_1 \dots h_n$ . Similarly, any degree  $r$  factor  $f$  of  $h$  can be written as  $f = f_1 \dots f_r$  where for each  $i$ ,  $f_i$  is the product of all the monic irreducible degree  $i$  factors of  $f$ , including multiplicities,  $f_i \in \mathcal{P}_i^+$  and  $f_i$  divides  $h_i$ .

Since each degree  $r$  factor  $f$  of  $h$  corresponds to a unique  $M = M(f) = (m_{ij}) \in \mathcal{M}(r)$ , it follows that  $r = \sum_i i \left( \sum_j j m_{ij} \right)$ . Since all the factors of  $f_i$  are degree  $i$  polynomials,  $f_i$  ‘corresponds’ to row  $i$  of the matrix  $M = M(f) = (m_{ij})$ . We define the matrix to which  $f_i$  corresponds as

$$M_i := M_i(M) = (m'_{kj}) = \begin{cases} m_{ij} & \text{if } k = i \\ 0 & \text{if } k \neq i. \end{cases} \quad (14)$$

Let  $r_i := r_i(M_i) := i \left( \sum_j j m_{ij} \right)$ . Then it follows that  $M_i \in \mathcal{M}(r_i)$ , and  $r_i = \deg(f_i)$ . Also,  $M = \sum_{i=1}^r M_i$ .

The problem of calculating the number of degree  $r$  factors of  $h$  corresponding to  $M$ , can be solved by calculating for each  $i$ , the number of monic degree  $r_i$  polynomials dividing  $h_i$  corresponding to  $M_i$ . We have

$$\alpha(h; r) = \sum_{M \in \mathcal{M}(r)} \alpha(h; r, M) = \sum_{M \in \mathcal{M}(r)} \left( \prod_{i=1}^r \alpha(h_i; r_i, M_i) \right). \quad (15)$$

We should note that  $\alpha(h_i; 0, M_i) = 1$  for all  $M_i \in \mathcal{M}(0)$ , all  $h_i \in \mathcal{P}_i^+$  and all  $i$  since the constant polynomial 1 divides  $h_i$ .

Also note that if the degree of  $h$  is less than  $r$  then  $h$  clearly has no degree  $r$  factors. Hence in this case we have, for all  $M \in \mathcal{M}(r)$ ,  $\alpha(h; r, M) = 0$  since for at least one  $i$ , we must have  $\alpha(h_i; r_i, M_i) = 0$ .

Below we define  $\tau$ -parameters and then determine  $\alpha(h_i; r_i, M_i)$  in terms of these parameters.

**Definition 4.5.** Let  $\tau(h; i, j)$  denote the number of distinct monic irreducible degree  $i$  factors of  $h$  that have multiplicity exactly  $j$ . Let  $\tau(h; i, j, +)$  be the

number of distinct monic irreducible degree  $i$  factors of  $h$  with multiplicity  $j$  or greater.

**Lemma 4.6.** For  $M = (m_{ij}) \in \mathcal{M}(r)$ ,

$$\alpha(h_i; r_i, M_i) = \prod_{j=1}^{\lfloor r/i \rfloor} \binom{\tau(h_i; i, j, +) - \sum_{k>j} m_{ik}}{m_{ij}}.$$

*Proof.* Set  $m_i := \sum_j m_{ij}$ . We describe the process of selecting the  $m_i$  pairwise distinct monic irreducible degree  $i$  factors of  $f$  for a fixed value of  $i$ . If  $m_i = 0$  there is nothing to do so assume that  $m_i > 0$  and let  $j(i, M)$  be the largest integer  $j$  such that  $m_{ij} > 0$ . Note that  $j(i, M) \leq r/i$  by Definition 4.4 (1). The  $m_{ij(i, M)}$  monic irreducible degree  $i$  factors of  $f$  with multiplicity  $j(i, M)$  can be chosen in  $\binom{\tau(h; i, j(i, M), +)}{m_{ij(i, M)}}$  ways. Once these are chosen we must not choose them again as factors to ensure they have the correct multiplicity in  $f$ . Thus for the next largest  $j$  such that  $m_{ij} > 0$ , we choose the  $m_{ij}$  monic irreducible degree  $i$  factors having multiplicity  $j$  from the remaining available  $\tau(h; i, j, +) - m_{ij(i, M)}$  irreducible factors of  $h_i$  having multiplicity at least  $j$ . In general, if  $j$  is such that we have already chosen the  $m_{ik}$  monic irreducible degree  $i$  factors of  $f$  of multiplicity  $k$  for all  $k > j$ , then we may choose the  $m_{ij}$  such factors of  $f$  of multiplicity  $j$  in exactly  $\binom{\tau(h; i, j, +) - \sum_{k>j} m_{ik}}{m_{ij}}$  ways. Hence the result

$$\alpha(h_i; r_i, M_i) = \prod_{j=1}^{j(i, M)} \binom{\tau(h; i, j, +) - \sum_{k>j} m_{ik}}{m_{ij}}.$$

□

We now define the function  $\Phi^p$  for a monic irreducible degree  $i$  polynomial  $p$  as

$$\Phi^p = \Phi^p(t, (s_{ij})_{j \geq 1}) := 1 + \sum_{j=1}^{\infty} \frac{s_{ij} t^{ij}}{\text{Cent}(i, j)}. \quad (16)$$

As with  $F^p$ ,  $\Phi^{p_1} = \Phi^{p_2}$  if  $p_1$  and  $p_2$  are monic, irreducible polynomials of the same degree. So for all  $i \in \mathbb{Z}^+$  we will define  $\Phi_i^+$  as the product of the  $\Phi^p$  for all monic irreducible degree  $i$  polynomials  $p$  with a nonzero constant term. Thus

$$\Phi_i^+ = \Phi_i^+(t, (s_{ij})_{j \geq 1}) := \left( 1 + \sum_{j=1}^{\infty} \frac{s_{ij} t^{ij}}{\text{Cent}(i, j)} \right)^{N^+(i, q)}. \quad (17)$$

**Lemma 4.7.** As a power series,  $\Phi_i^+$  satisfies:

$$(1). \quad \Phi_i^+ = \sum_{h_i \in \mathcal{P}_i^+} \frac{(\prod_j s_{ij}^{\tau(h_i; i, j)}) t^{\deg(h_i)}}{\text{Cent}(h_i)}, \quad \text{and}$$

$$(2). \prod_{i=1}^{\infty} \Phi_i^+ = \sum_{h \in \mathcal{P}^+} \frac{(\prod_{i,j} s_{ij}^{\tau(h;i,j)}) t^{\deg(h)}}{Cent(h)},$$

where  $\mathcal{P}^+$  and  $\mathcal{P}_i^+$  are as in Definition 3.1.

*Proof.* (1). Every polynomial in  $\mathcal{P}_i^+$  corresponds to a summand of  $\Phi_i^+$ , namely each  $h_i \in \mathcal{P}_i^+$  is a product of  $p^{j(p)}$  (where  $j(p) \geq 0$ ) over all monic degree  $i$  irreducible polynomials  $p$ , and  $\deg(h_i) = \sum_p ij(p)$ . The corresponding summand of  $\Phi_i^+$  is  $\frac{(\prod_p s_{ij(p)}) t^{\deg(h_i)}}{Cent(h_i)}$  and this is obtained by choosing the term corresponding to  $p^{j(p)}$ , which is  $\frac{s_{ij(p)} t^{ij(p)}}{Cent(i,j(p))}$  if  $j(p) \geq 1$  and 1 if  $j(p) = 0$ . The denominator of the summand corresponding to  $h_i$  is  $Cent(h_i)$  since  $\prod_p Cent(i, j(p)) = Cent(\prod_p p^{j(p)}) = Cent(h_i)$  by Lemma 2.12. The exponent of  $t$  is  $\sum_p ij(p) = \deg(h_i)$ . For each monic irreducible degree  $i$  polynomial with multiplicity  $j$ , the corresponding  $\Phi^p$  contains an  $s_{ij}$  term, so the exponent of  $s_{ij}$  in the summand corresponding to  $h_i$  is the number of monic irreducible degree  $i$  factors  $p$  of  $h_i$  such that  $j(p) = j$ , that is  $\tau(h_i; i, j)$ . Hence

$$\Phi_i^+ = \sum_{h_i \in \mathcal{P}_i^+} \frac{(\prod_j s_{ij}^{\tau(h_i;i,j)}) t^{\deg(h_i)}}{Cent(h_i)}.$$

(2).  $\prod_i \Phi_i^+$  is simply the product over all monic irreducible polynomials with nonzero constant terms as opposed to the computation in case (1) where we just chose those irreducible polynomials with degree  $i$ . Hence every polynomial in  $\mathcal{P}^+$  will correspond to a certain term in  $\prod_i \Phi_i^+$ . By part (1) the term corresponding to a polynomial  $h$  is

$$\frac{(\prod_{i,j} s_{ij}^{\tau(h;i,j)}) t^{\deg(h)}}{Cent(h)}$$

where now we have  $s_{ij}$  present for each  $i$  such that  $\tau(h; i, j) > 0$ . Hence we get our result

$$\prod_{i=1}^{\infty} \Phi_i^+ = \sum_{h \in \mathcal{P}^+} \frac{(\prod_{i,j} s_{ij}^{\tau(h;i,j)}) t^{\deg(h)}}{Cent(h)}.$$

□

Note again that for each  $i \in \mathbb{Z}^+$  the only occurrences of  $s_{ij}$  for any  $j$  are in  $\Phi_i^+$ . So if we partially differentiate  $\Phi^+$  with respect to  $s_{ij}$  for some  $j$  we get the same answer as if we partially differentiated  $\Phi_i^+$  with respect to  $s_{ij}$  and multiplied by the product of the remaining  $\Phi_{i'}^+$ ,  $i' \neq i$ .

We will perform a series of operations on each  $\Phi_i^+$  in turn so that after this procedure for  $\Phi_i^+$  the coefficient of the term corresponding to a polynomial  $h_i \in \mathcal{P}_i^+$  and an  $M \in \mathcal{M}(r)$  will be  $\alpha(h_i; r_i, M_i) / Cent(h_i)$ .

**Definition 4.8.** For  $i \geq 1$  and  $M \in \mathcal{M}(r)$ , let

$$\Phi_{i,M,\alpha}^+ = \Phi_{i,M,\alpha}^+(t) := \sum_{h_i \in \mathcal{P}_i^+} \alpha(h_i; r_i, M_i) \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)},$$

where  $r_i = i \sum_j j m_{ij}$ .

Note that, if  $r_i = 0$  (which holds in particular if  $i > r$ ) then  $\alpha(h_i; r_i, M_i) = 1$  for each  $h_i \in \mathcal{P}_i^+$ , and hence in this case  $\Phi_{i,M,\alpha}^+ = F_i$  as defined in (7). In this case the leading term of  $\Phi_{i,M,\alpha}^+$  is 1 and corresponds to the constant polynomial  $1 \in \mathcal{P}_i^+$ .

The product of  $\Phi_{i,M,\alpha}^+$  over all  $i$  is a power series such that the coefficient of the term corresponding to a polynomial  $h \in \mathcal{P}^+$  is  $\alpha(h; r, M)/\text{Cent}(h)$ . Then summing over all  $M \in \mathcal{M}(r)$  will produce a series such that the coefficient of the term corresponding to  $h$  is  $\alpha(h; r)/\text{Cent}(h)$ , that is  $C_{\text{GL},r}(t)$ . We prove this now.

**Lemma 4.9.**  $C_{\text{GL},r}(t) = \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+(t).$

*Proof.* From Definition 4.8 we have that

$$\Phi_{i,M,\alpha}^+ = \sum_{h_i \in \mathcal{P}_i^+} \alpha(h_i; r_i, M_i) \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)}.$$

Recall from Definition 3.1, that  $\mathcal{P}_i^+$  is the set containing the constant polynomial 1 along with all monic polynomials that have a nonzero constant term and whose only irreducible factors have degree  $i$ . Each  $h \in \mathcal{P}^+$ , that is, each monic polynomial with nonzero constant term, has a unique factorisation,  $h = \prod_i h_i$ , where  $h_i \in \mathcal{P}_i^+$  for each  $i$ . Using this notation we have

$$\prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+ = \sum_{h \in \mathcal{P}^+} \prod_{i=1}^{\infty} \frac{\alpha(h_i; r_i, M_i) t^{\deg(h_i)}}{\text{Cent}(h_i)}.$$

By Equation 15 we know that  $\prod_i \alpha(h_i; r_i, M_i) = \alpha(h; r, M)$ . By Lemma 2.12 we know that  $\prod_i \text{Cent}(h_i) = \text{Cent}(h)$  since the  $h_i$  are pairwise coprime and we know that each polynomial  $h_i$  corresponds to the  $t$ -power  $t^{\deg(h_i)}$  so  $\prod_i t^{\deg(h_i)} = t^{\deg(h)}$ . Thus

$$\prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+ = \sum_{h \in \mathcal{P}^+} \frac{\alpha(h; r, M) t^{\deg(h)}}{\text{Cent}(h)}.$$

Now we want to sum over all  $M \in \mathcal{M}(r)$ . The sum over  $h \in \mathcal{P}^+$  and the sum over  $M \in \mathcal{M}(r)$  can be interchanged so that

$$\begin{aligned}
\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+ &= \sum_{M \in \mathcal{M}(r)} \sum_{h \in \mathcal{P}^+} \frac{\alpha(h; r, M) t^{\deg(h)}}{Cent(h)} \\
&= \sum_{h \in \mathcal{P}^+} \frac{\left( \sum_{M \in \mathcal{M}(r)} \alpha(h; r, M) \right) t^{\deg(h)}}{Cent(h)}.
\end{aligned}$$

By Lemma 15, we know that  $\sum_{M \in \mathcal{M}(r)} \alpha(h; r, M) = \alpha(h; r)$  and by using Equation (13) we have

$$\begin{aligned}
\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+ &= \sum_{h \in \mathcal{P}^+} \frac{\alpha(h; r) t^{\deg(h)}}{Cent(h)} \\
&= \sum_{n=0}^{\infty} c_{\text{GL},r}(n) t^n \\
&= \sum_{n=r}^{\infty} c_{\text{GL},r}(n) t^n \\
&= C_{\text{GL},r}(t)
\end{aligned}$$

since  $c_{\text{GL},r}(n)$  is trivially evaluated to 0 for  $n < r$ .  $\square$

We now describe the procedure to produce  $\Phi_{i,M,\alpha}^+$  for a fixed  $r \in \mathbb{Z}^+$ , a fixed  $M = (m_{ij}) \in \mathcal{M}(r)$  and a fixed  $i < r$ .

**Procedure:** PHIALPHA ( $r, M, i$ )

Input:  $r = \dim(U)$ ,  $M \in \mathcal{M}(r)$ ,  $i \in \mathbb{Z}^+$ .

Output  $\Phi_{i,M,\alpha}^+(t)$

Operations

Set  $\Psi := \Phi_i^+$ ,  $s_{i0} := 1$  and  $k := \lfloor \frac{r}{i} \rfloor$ .

For  $j > k$  assign  $s_{ij} := s_{ik}$  in  $\Psi$ .

While  $k > 0$

Set  $\Psi := \frac{1}{m_{ik}!} \frac{\partial^{m_{ik}}}{\partial s_{ik}^{m_{ik}}} \Psi$ .

Set  $s_{ik} := s_{i(k-1)}$  and  $k := k - 1$ .

Return  $\Psi$

**Lemma 4.10.** (1) The procedure  $\text{PHIALPHA}(r, M, i)$  correctly returns  $\Phi_{i,M,\alpha}^+$ ;  
(2) The smallest  $j$  such that  $t^j$  has nonzero coefficient in the power series expansion of  $\Phi_{i,M,\alpha}^+(t)$  is  $j = r_i$ .  
(3) Moreover, if  $i \sum_j j m_{ij} = r_i = 0$  then  $\Phi_{i,M,\alpha}^+(t) = F_i(t)$ .

*Proof.* The initial value of  $k$  is 0 if and only if  $i > r$  and in this case the procedure returns  $\Phi_i^+|_{(s_{ij}=1 \text{ for all } j)}$ . This power series is equal to  $F_i$ , as defined in Equation (7), and is correct since we do not wish to choose any polynomials of this degree (since they cannot divide  $f$ ). As noted after Definition 4.8, in this case  $r_i = 0$ ,  $F_i(t) = \Phi_{i,M,\alpha}^+(t)$  and the leading term is 1. Thus the assertions (1) – (3) follow in this case.

Assume now that  $i \leq r$ . We first prove part (1). At the start of the procedure  $\Psi := \Phi_i^+$ . The value  $k = \lfloor r/i \rfloor \geq 1$  is the highest possible multiplicity of a degree  $i$  factor we could choose to be a factor of  $f$ .

Let  $j_0 = \lfloor r/i \rfloor$ . Start by making  $s_{ij}$  equal  $s_{ij_0}$  in  $\Psi$  for all  $j > j_0$ . After this the exponent of  $s_{ij_0}$  is the number of irreducible degree  $i$  factors of  $h$  that have multiplicity at least  $j_0$ , that is  $\tau(h; i, j_0, +)$ . At the beginning of the while loop we have

$$\Psi = \sum_{h_i \in \mathcal{P}_i^+} \left( \left( \prod_{j < j_0} s_{ij}^{\tau(h_i; i, j)} \right) s_{ij_0}^{\tau(h_i; i, j_0, +)} \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)} \right).$$

We repeat the steps in the while loop  $j_0$  times. After performing the partial differentiation step the first time (for  $k = j_0$ ), Lemma 4.3 implies we get

$$\Psi = \sum_{h_i \in \mathcal{P}_i^+} \left( \left( \prod_{j < j_0} s_{ij}^{\tau(h_i; i, j)} \right) s_{ij_0}^{\tau(h_i; i, j_0, +) - m_{ij_0}} \binom{\tau(h_i; i, j_0, +)}{m_{ij_0}} \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)} \right).$$

The  $s_{ij}$  for  $j < j_0$  remain, while the power of  $s_{ij_0}$  is reduced by  $m_{ij_0}$ . The  $\binom{\tau(h_i; i, j_0, +)}{m_{ij_0}}$  appears as shown in Lemma 4.3. If  $j_0 = 1$  then we assign  $s_{i1} := 1$  and we return

$$\Psi = \sum_{h_i \in \mathcal{P}_i^+} \binom{\tau(h_i; i, j_0, +)}{m_{ij_0}} \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)}.$$

By Lemma 4.6, if  $j_0 = 1$  then  $\binom{\tau(h_i; i, j_0, +)}{m_{ij_0}} = \alpha(h_i; r_i, M_i)$  and hence  $\Psi = \Phi_{i,M,\alpha}^+$  is correctly returned.

Now assume that  $j_0 > 1$  so that we assign  $s_{ij_0}$  to equal  $s_{i(j_0-1)}$ . The new exponent on  $s_{i(j_0-1)}$  will be  $\tau(h_i; i, (j_0-1), +) - \sum_{j > j_0} m_{ij}$ . For a general  $k$  the while loop proceeds as follows. After differentiating, by Lemma 4.3 the term corresponding to  $h_i \in \mathcal{P}_i^+$  is multiplied by  $\binom{\tau(h_i; i, k, +) - \sum_{j > k} m_{ij}}{m_{ik}}$ , and the  $s_{ij}$  are modified. Hence after running the while loop for  $k = j_0, j_0 - 1, \dots, 1$  we will have produced the coefficient

$$\prod_{j=1}^{j_0} \binom{\tau(h_i; i, j, +) - \sum_{k>j} m_{ik}}{m_{ij}}$$

and the function returned is

$$\Psi = \sum_{h_i \in \mathcal{P}_i^+} \prod_{j=1}^{j_0} \binom{\tau(h_i; i, j, +) - \sum_{k>j} m_{ik}}{m_{ij}} \frac{t^{\deg(h_i)}}{Cent(h_i)}.$$

By Lemma 4.6 we see that we have correctly returned

$$\Psi = \Psi(t) = \sum_{h_i \in \mathcal{P}_i^+} \alpha(h_i; r_i, M_i) \frac{t^{\deg(h_i)}}{Cent(h_i)} = \Phi_{i,M,\alpha}^+(t).$$

Thus part (1) is proved. In particular, if  $r_i = 0$  then  $\alpha(h_i; r_i, M_i) = 1$  so that  $\Phi_{i,M,\alpha}^+ = F_i$ , so part (3) holds.

Let  $l < r_i = i \sum_j j m_{ij}$ . If  $h_i \in \mathcal{P}_i^+$  and  $\deg(h_i) = l$ , then  $h_i$  has no factors of degree  $r_i$ , so  $\alpha(h_i; r_i, M_i) = 0$ . Thus the coefficient of  $t^l$  in  $\Phi_{i,M,\alpha}^+(t)$  is zero. On the other hand if  $l = r_i$  then there exists at least one  $h_i \in \mathcal{P}_i^+$  of degree  $r_i$  (even if  $r_i = 0$ ) such that  $h_i$  has  $m_{ij}$  monic irreducible degree  $i$  factors of multiplicity  $j$  for each  $j$ . For each such  $h_i$ ,  $\alpha(h_i; r_i, M_i) = 1$  and so the coefficient of  $t^{r_i}$  is nonzero. Thus part (2) is proved.  $\square$

We now give an alternative expression for determining the generating function  $C_{GL,r}(t)$  that involves Wall's function  $C_{GL}(t)$  defined in Equation (8). This expression will be used later to study the asymptotic properties of  $c_{GL,r}(n)$ .

**Theorem 4.11.** *For each  $i$ , let  $m_i := \sum_j m_{ij}$ . Then*

$$C_{GL,r}(t) = C_{GL}(t) \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}^+(t)$$

where  $\phi_{i,M}^+(t) = m_i! \binom{N^+(i, q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{(tq^{-1})^{ijm_{ij}}}{m_{ij}!(1 - q^{-i} + t^i q^{-2i})^{m_{ij}}}$  and  $C_{GL}(t) = \prod_i F_i$  as in Equation (8). Moreover, if  $i > r$  then  $\phi_{i,M}^+(t) = 1$  for all  $M \in \mathcal{M}(r)$ .

*Proof.* By Lemma 4.9,

$$C_{GL,r}(t) = \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^{\infty} \Phi_{i,M,\alpha}^+(t)$$

and by (8)

$$C_{GL}(t) = \prod_{i=1}^{\infty} F_i(t).$$

Also in Lemma 4.10 (2), for  $i > r$  we have  $\Phi_{i,M,\alpha}^+(t) = F_i(t)$ . Then it follows that

$$C_{GL,r}(t) = C_{GL}(t) \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \frac{\Phi_{i,M,\alpha}^+(t)}{F_i(t)}$$

where by (7),  $F_i(t) = \left(1 + \sum_{j=1}^{\infty} \frac{t^{ij}}{Cent(i,j)}\right)^{N^+(i,q)}$  and by Lemma 2.11,  $Cent(i,j) = q^{ij}(1 - q^{-i})$ .

We consider the details of the procedure PHIALPHA( $r, M, i$ ) that constructs  $\Phi_{i,M,\alpha}^+(t)$ . Firstly the procedure sets  $\Psi := \Phi_i^+$  and assign  $k := \lfloor r/i \rfloor$ . Then the  $s_{ij}$  are assigned to  $s_{ik}$  for all  $j > k$  so that  $\Psi$  becomes

$$\left(1 + \sum_{j=1}^{k-1} \frac{s_{ij}t^{ij}}{q^{ij}(1 - q^{-i})} + \sum_{j=k}^{\infty} \frac{s_{ik}t^{ij}}{q^{ij}(1 - q^{-i})}\right)^{N^+(i,q)} = \Psi_0^{N^+(i,q)}$$

where  $\Psi_0$  is given by  $1 + \sum_{j=1}^{k-1} \frac{s_{ij}t^{ij}}{q^{ij}(1 - q^{-i})} + \sum_{j=k}^{\infty} \frac{s_{ik}t^{ij}}{q^{ij}(1 - q^{-i})}$ .

In the first run of the while loop, we partially differentiate  $\Psi$  with respect to  $s_{ik}$ ,  $m_{ik}$  times, and divide by  $m_{ik}!$ . Suppose first that  $m_{ik} > 0$ . Then, by the chain rule, this will produce

$$N^+(i,q)(N^+(i,q) - 1) \dots (N^+(i,q) - m_{ik} + 1) = \prod_{0 \leq u < m_{ik}} (N^+(i,q) - u).$$

The first partial derivative of  $\Psi_0$  is  $\sum_{j=k}^{\infty} \frac{t^{ij}}{q^{ij}(1 - q^{-i})}$  which is a geometric progression equal to  $\frac{t^{ik}}{q^{ik}(1 - q^{-i})(1 - t^i/q^i)}$ . Thus after performing the partial differentiation and division by  $m_{ik}!$  we obtain

$$\left(\prod_{0 \leq u < m_{ik}} (N^+(i,q) - u)\right) \frac{1}{m_{ik}!} \left(\frac{t^{ik}}{q^{ik}(1 - q^{-i})(1 - t^i/q^i)}\right)^{m_{ik}} \Psi_0^{N^+(i,q) - m_{ik}}.$$

The final step in the while loop is to change  $s_{ik}$  into  $s_{i(k-1)}$  and hence  $\Psi_0$  becomes

$$1 + \sum_{j=1}^{k-2} \frac{s_{ij}t^{ij}}{q^{ij}(1 - q^{-i})} + \sum_{j=k-1}^{\infty} \frac{s_{i(k-1)}t^{ij}}{q^{ij}(1 - q^{-i})}.$$

As the procedure prescribes, we now repeat the while loop using  $k - 1$  in place of  $k$ . After performing the while loop  $\lfloor r/i \rfloor$  times we will have partially differentiated  $m_i$  times. After all runs of the while loop,  $\Psi$  will equal



$$\left( \prod_{0 \leq u < m_i} (N^+(i, q) - u) \right) \prod_{j=1}^{\lfloor r/i \rfloor} \frac{1}{m_{ij}!} \left( \frac{t^{ij}}{q^{ij}(1-q^{-i})(1-t^i q^{-i})} \right)^{m_{ij}} \hat{\Psi}_0^{N^+(i, q) - m_i}$$

where  $\hat{\Psi}_0$  is equal to the function  $\Psi_0$  with all the  $s_{ij}$  set to 1. We make note of the fact that we only need to take the product over  $j$  from 1 to  $\lfloor r/i \rfloor$  because for  $j > \lfloor r/i \rfloor$  we have that  $m_{ij} = 0$  and hence  $\frac{1}{m_{ij}!} \left( \frac{t^{ij}}{q^{ij}(1-q^{-i})(1-t^i q^{-i})} \right)^{m_{ij}} = 1$ . This final expression for  $\Psi$  equals  $\Phi_{i, M, \alpha}^+(t)$ , by Lemma 4.10.

Finally observe that

$$\begin{aligned} \hat{\Psi}_0(t) &= 1 + \sum_{j=1}^{\infty} \frac{t^{ij}}{q^{ij}(1-q^{-i})} \\ &= 1 + \frac{t^i}{q^i(1-q^{-i})(1-t^i q^{-i})} \end{aligned}$$

and hence, by Equation 7, substituting in the values of  $Cent(i, j)$ , we have  $F_i(t) = \hat{\Psi}_0(t)^{N^+(i, q)}$ . Since  $m_i$  equals  $\sum_j m_{ij}$  and  $m_i! \binom{N^+(i, q)}{m_i}$  is equal to  $\prod_{0 \leq u < m_i} (N^+(i, q) - u)$ , it follows that  $\frac{\Phi_{i, M, \alpha}^+(t)}{F_i(t)}$  is equal to

$$\begin{aligned} m_i! \binom{N^+(i, q)}{m_i} \frac{1}{\hat{\Psi}_0(t)^{m_i}} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{1}{m_{ij}!} \left( \frac{t^{ij}}{q^{ij}(1-q^{-i})(1-t^i q^{-i})} \right)^{m_{ij}} &= \\ m_i! \binom{N^+(i, q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{1}{m_{ij}!} \left( \frac{1}{(1 + \frac{t^i}{q^i(1-q^{-i})(1-t^i/q^i)})} \frac{t^{ij}}{q^{ij}(1-q^{-i})(1-t^i/q^i)} \right)^{m_{ij}}. \end{aligned}$$

Rearranging gives

$$\begin{aligned} \frac{\Phi_{i, M, \alpha}^+(t)}{F_i(t)} &= m_i! \binom{N^+(i, q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{1}{m_{ij}!} \left( \frac{t^{ij} q^{-ij}}{(1-q^{-i})(1-t^i q^{-i}) + t^i q^{-i}} \right)^{m_{ij}} \\ &= m_i! \binom{N^+(i, q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{1}{m_{ij}!} \left( \frac{t^{ij} q^{-ij}}{1 - q^{-i} + t^i q^{-2i}} \right)^{m_{ij}} \\ &= m_i! \binom{N^+(i, q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{(t q^{-1})^{ij m_{ij}}}{m_{ij}! (1 - q^{-i} + t^i q^{-2i})^{m_{ij}}}. \\ &= \phi_{i, M}^+(t). \end{aligned}$$

□

## 4.2 The Limiting Proportion

Now that we have a formula for the generating function,  $C_{\text{GL},r}(t)$ , for any  $r$ , we will prove that its coefficients converge and find their limiting value. We use the expression for  $\phi_{i,M}^+(t)$  from Theorem 4.11.

**Lemma 4.12.** *Let  $M = (m_{ij}) \in \mathcal{M}(r)$ . Then for any  $i \leq r$ , the power series expansion of*

$$\phi_{i,M}^+(t) = m_i! \binom{N^+(i,q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{(tq^{-1})^{ijm_{ij}}}{m_{ij}!(1 - q^{-i} + t^i q^{-2i})^{m_{ij}}},$$

is convergent for  $|t| < q(q^i - 1)^{1/i}$ .

*Proof.* We first express  $\phi_{i,M}^+(t)$  as follows:

$$\phi_{i,M}^+(t) = m_i! \binom{N^+(i,q)}{m_i} t^{r_i} q^{-r_i} (1 - q^{-i} + t^i q^{-2i})^{-m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \left( \frac{1}{m_{ij}!} \right).$$

Clearly the polynomial  $m_i! \binom{N^+(i,q)}{m_i} t^{r_i} q^{-r_i} \prod_{j=1}^{\lfloor r/i \rfloor} \left( \frac{1}{m_{ij}!} \right)$  is convergent for all  $t$ , but  $(1 - q^{-i} + t^i q^{-2i})^{-m_i}$  needs further inspection. We need to find the radius of convergence of the power series for  $(1 - q^{-i} + t^i q^{-2i})^{-1}$ . Factorisation gives

$$\frac{1}{1 - q^{-i} + t^i q^{-2i}} = \frac{1}{1 - q^{-i}} \left( \frac{1}{1 + \frac{t^i q^{-2i}}{1 - q^{-i}}} \right).$$

Now  $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$  is convergent if and only if  $|x| < 1$  and hence the power series for the displayed function is convergent if  $|\frac{t^i q^{-2i}}{1 - q^{-i}}| < 1$ , that is,  $|t^i| < q^i(q^i - 1)$ . This is equivalent to  $|t| < q(q^i - 1)^{1/i}$ .  $\square$

**Corollary 4.13.** *The power series expansion of  $\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}^+(t)$  is convergent for  $|t| < q(q - 1)$ .*

*Proof.* By Lemma 4.12, the power series expansion of  $\phi_{i,M}^+(t)$  is convergent for  $|t| < q(q^i - 1)^{1/i}$  for all  $i \leq r$  and any  $M \in \mathcal{M}(r)$ . Hence  $\prod_i \phi_{i,M}^+(t)$  is convergent for  $|t|$  less than the minimum of  $q(q^i - 1)^{1/i}$  over  $i = 1, \dots, r$ . That is to say, for  $|t| < q(q - 1)$ . Hence we have our result since  $\mathcal{M}(r)$  is finite.  $\square$

Despite the above corollary,  $C_{\text{GL},r}(t)$  is only convergent for  $|t| < 1$  because  $C_{\text{GL}}(t)$  is only convergent for  $|t| < 1$  (see our comment before Theorem 3.4).

The next theorem gives us a definite formula for the limit of  $c_{\text{GL},r}(n)$  as  $n$  tends to infinity.

**Theorem 4.14.** For  $r \in \mathbb{Z}^+$ ,  $c_{\text{GL},r}(\infty) := \lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  exists and satisfies

$$c_{\text{GL},r}(\infty) = \frac{1 - q^{-5}}{1 + q^{-3}} \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}^+(1),$$

where  $\phi_{i,M}^+(t)$  is defined as in Theorem 4.11. Moreover, for any  $d$  such that  $1 < d < q(q-1)$ ,  $|c_{\text{GL},r}(n) - c_{\text{GL},r}(\infty)| = O(d^{-n})$ .

*Proof.* From [22],  $(1-t)C_{\text{GL}}(t)$  is convergent for  $|t| < q^2$  and by Corollary 4.13,  $\sum_{M \in \mathcal{M}(r)} \prod_i \phi_{i,M}^+(t)$  is convergent for  $|t| < q(q-1)$ . Hence  $(1-t)C_{\text{GL},r}(t)$  is convergent for  $|t| < q(q-1)$ .

By Lemma 3.3,  $(1-t)C_{\text{GL},r}(t)$  evaluated at  $t = 1$  will give us the limit of  $c_{\text{GL},r}(n)$  as  $n \rightarrow \infty$ . By [22, Equation 6.23] we know that  $(1-t)C_{\text{GL}}(t)$  evaluated at  $t = 1$  is

$$\frac{1 - q^{-5}}{1 + q^{-3}}.$$

Thus

$$\lim_{n \rightarrow \infty} c_{\text{GL},r}(n) = \frac{1 - q^{-5}}{1 + q^{-3}} \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}^+(1).$$

We showed above that  $(1-t)C_{\text{GL},r}(t)$  is convergent for  $|t| < q(q-1)$ , so the final assertion stating the rate of convergence follows by Lemma 3.3.  $\square$

We note that the existence of the limit  $c_{\text{GL},r}(\infty)$  and the convergence rate of  $c_{\text{GL},r}(n)$  asserted in Theorem 1.1 follow from Theorem 4.14.

We now evaluate  $\phi_{i,M}^+(1)$  for all  $i$  and determine the first few terms of the power series expansion. This will aid us in proving Theorem 4.22 which determines the first nontrivial term in the power series expansion for  $c_{\text{GL},r}(\infty)$ .

**Lemma 4.15.** Let  $M = (m_{ij}) \in \mathcal{M}(r)$  and set  $m_i = \sum_j m_{ij}$  for each  $i$ . Then

with the notation of Theorem 4.11 we get

$$\begin{aligned}
\phi_{1,M}^+(1) &= \frac{q^{m_1 - \sum j m_{1j}}}{\prod_j m_{1j}!} \left( 1 + \left( -\frac{m_1^2}{2} + \frac{m_1}{2} \right) q^{-1} \right. \\
&\quad \left. + \left( \frac{m_1^4}{8} - \frac{5m_1^3}{12} - \frac{m_1^2}{8} - \frac{7m_1}{12} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{2,M}^+(1) &= \frac{q^{2m_2 - 2 \sum j m_{2j}}}{2^{m_2} \prod_j m_{2j}!} \left( 1 - m_2 q^{-1} + \left( -\frac{m_2^2}{2} + \frac{3m_2}{2} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{3,M}^+(1) &= \frac{q^{3m_3 - 3 \sum j m_{3j}}}{3^{m_3} \prod_j m_{3j}!} (1 - m_3 q^{-2} + O(q^{-3})), \\
\phi_{4,M}^+(1) &= \frac{q^{4m_4 - 4 \sum j m_{4j}}}{4^{m_4} \prod_j m_{4j}!} (1 - m_4 q^{-2} + O(q^{-3})), \\
\phi_{i,M}^+(1) &= \frac{q^{im_i - i \sum j m_{ij}}}{i^{m_i} \prod_j m_{ij}!} (1 + O(q^{-3})) \quad \text{for } i \geq 5.
\end{aligned}$$

*Proof.* We consider  $\phi_{i,M}^+(1)$  separately for  $i = 1, 2, 3, 4$  before looking at the general case  $i \geq 5$ .

**Case:  $i = 1$**

From the definition of  $\phi_{i,M}^+(t)$  in Theorem 4.11,

$$\phi_{1,M}^+(1) = m_1! \binom{N^+(1, q)}{m_1} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{q^{-j m_{1j}}}{m_{1j}! (1 - q^{-1} + q^{-2})^{m_{1j}}}.$$

We consider  $\phi_{1,M}^+(1)$  in three parts. First of all, noting that  $m_1 \leq N^+(1, q) = q - 1$  since  $m_1$  is the number of distinct linear factors of a polynomial  $f$ , we have  $m_1! \binom{N^+(1, q)}{m_1} = (q - 1)(q - 2) \dots (q - m_1)$ , which equals

$$\begin{aligned}
&q^{m_1 - \frac{m_1(m_1+1)}{2}} q^{m_1-1} + \left( \frac{m_1^4}{8} + \frac{m_1^3}{12} - \frac{m_1^2}{8} - \frac{m_1}{12} \right) q^{m_1-2} + O(q^{m_1-3}) \\
&= q^{m_1} \left( 1 - \left( \frac{m_1^2}{2} + \frac{m_1}{2} \right) q^{-1} + \left( \frac{m_1^4}{8} + \frac{m_1^3}{12} - \frac{m_1^2}{8} - \frac{m_1}{12} \right) q^{-2} + O(q^{-3}) \right).
\end{aligned} \tag{18}$$

where the coefficient of  $q^{m_1-2}$  was computed in detail in Lemma 2.5.1 of [4].

Secondly,

$$\prod_j \frac{q^{-j m_{1j}}}{m_{1j}!} = \frac{q^{-\sum j m_{1j}}}{\prod_j m_{1j}!}.$$

Finally, since  $\frac{1}{(1-x)^m} = \sum_{k \geq 0} \binom{m+k-1}{k} x^k$ , making a Taylor expansion we get

$$\begin{aligned}
\prod_j \frac{1}{(1 - q^{-1} + q^{-2})^{m_{1j}}} &= \frac{1}{(1 - q^{-1} + q^{-2})^{m_1}} \\
&= 1 + m_1(q^{-1} - q^{-2}) + \binom{m_1+1}{2}(q^{-1} - q^{-2})^2 + \dots \\
&= 1 + m_1 q^{-1} + \frac{m_1(m_1-1)}{2} q^{-2} + O(q^{-3}).
\end{aligned}$$

Multiplying together the three parts and collecting terms gives us  $\phi_{1,M}^+(1)$  equal to

$$\frac{q^{m_1 - \sum j m_{1j}}}{\prod_j m_{1j}!} \times \left( 1 + \left( -\frac{m_1^2}{2} + \frac{m_1}{2} \right) q^{-1} + \left( \frac{m_1^4}{8} - \frac{5m_1^3}{12} - \frac{m_1^2}{8} - \frac{7m_1}{12} \right) q^{-2} + O(q^{-3}) \right).$$

**Case:  $i = 2$**

From the definition of  $\phi_{i,M}^+(t)$  in Theorem 4.11,

$$\phi_{2,M}^+(1) = m_2! \binom{N(2,q)}{m_2} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{q^{-2jm_{2j}}}{m_{2j}!(1 - q^{-2} + q^{-4})^{m_{2j}}}.$$

Note that  $N(2, q) = N^+(2, q) = \frac{q^2 - q}{2}$ . We consider  $\phi_{2,M}^+(1)$  in three parts. First of all,

$$\begin{aligned} m_2! \binom{N(2,q)}{m_2} &= \left( \frac{1}{2}(q^2 - q) \right) \left( \frac{1}{2}(q^2 - q) - 1 \right) \dots \left( \frac{1}{2}(q^2 - q) - m_2 + 1 \right) \\ &= \frac{1}{2^{m_2}} (q^2 - q)(q^2 - q - 2) \dots (q^2 - q - 2m_2 + 2) \\ &= \frac{1}{2^{m_2}} \left( q^{2m_2} - m_2 q^{2m_2-1} - \frac{m_2(m_2-1)}{2} q^{2m_2-2} + O(q^{2m_2-3}) \right) \\ &= \frac{q^{2m_2}}{2^{m_2}} \left( 1 - m_2 q^{-1} + \left( -\frac{m_2^2}{2} + \frac{m_2}{2} \right) q^{-2} + O(q^{-3}) \right). \end{aligned} \tag{19}$$

The coefficient of  $q^{2m_2-2}$  arises from summing together  $\frac{m_2(m_2-1)}{2}$ , which is the number of ways to choose  $q^2$  from  $m_2 - 2$  terms and  $-q$  from two terms above, with  $-m_2(m_2 - 1)$ , which is the coefficient obtained by choosing  $q^2$  from  $m_2 - 1$  terms and a nonzero constant term from one term above.

The second part is

$$\prod_j \frac{q^{-2jm_{2j}}}{m_{2j}!} = \frac{q^{-2 \sum j m_{2j}}}{\prod_j m_{2j}!}.$$

Finally the Taylor expansion of

$$\begin{aligned} \prod_j \frac{1}{(1 - q^{-2} + q^{-4})^{m_{2j}}} &= \frac{1}{(1 - q^{-2} + q^{-4})^{m_2}} \\ &= 1 + m_2(q^{-2} - q^{-4}) + \binom{m_2+1}{2}(q^{-2} - q^{-4})^2 + \dots \\ &= 1 + m_2 q^{-2} + O(q^{-4}). \end{aligned}$$

Multiplying together the three parts and collecting terms gives us  $\phi_{2,M}^+(1)$  equal to

$$\frac{q^{2m_2 - 2 \sum j m_{2j}}}{2^{m_2} \prod_j m_{2j}!} \left( 1 - m_2 q^{-1} + \left( -\frac{m_2^2}{2} + \frac{3m_2}{2} \right) q^{-2} + O(q^{-3}) \right).$$

**Case:  $i = 3$**

From the definition of  $\phi_{i,M}^+(t)$  in Theorem 4.11,

$$\phi_{3,M}^+(1) = m_3! \binom{N(3,q)}{m_3} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{q^{-3jm_{3j}}}{m_{3j}!(1 - q^{-3} + q^{-6})^{m_{3j}}}.$$

Note that  $N(3,q) = N^+(3,q) = \frac{q^3 - q}{3}$ . We consider  $\phi_{3,M}^+(1)$  in three parts. First of all,

$$\begin{aligned} m_3! \binom{N(3,q)}{m_3} &= \left(\frac{1}{3}(q^3 - q)\right) \left(\frac{1}{3}(q^3 - q) - 1\right) \dots \left(\frac{1}{3}(q^3 - q) - m_3 + 1\right) \\ &= \frac{1}{3^{m_3}} (q^{3m_3} - m_3 q^{3m_3-2} + O(q^{3m_3-3})) \\ &= \frac{q^{3m_3}}{3^{m_3}} (1 - m_3 q^{-2} + O(q^{-3})). \end{aligned} \quad (20)$$

Secondly,

$$\prod_j \frac{q^{-3jm_{3j}}}{m_{3j}!} = \frac{q^{-3 \sum j m_{3j}}}{\prod_j m_{3j}!}.$$

Finally  $\prod_j \frac{1}{(1 - q^{-3} + q^{-6})^{m_{3j}}} = \frac{1}{(1 - q^{-3} + q^{-6})^{m_3}} = 1 + O(q^{-3})$ . Multiplying together the three parts and collecting terms gives us  $\phi_{3,M}^+(1)$  equal to

$$\frac{q^{3m_3-3 \sum j m_{3j}}}{3^{m_3} \prod_j m_{3j}!} (1 - m_3 q^{-2} + O(q^{-3})).$$

**Case:  $i = 4$**

From the definition of  $\phi_{i,M}^+(t)$  in Theorem 4.11,

$$\phi_{4,M}^+(1) = m_4! \binom{N(4,q)}{m_4} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{q^{-4jm_{4j}}}{m_{4j}!(1 - q^{-4} + q^{-8})^{m_{4j}}}.$$

Note that  $N(4,q) = N^+(4,q) = \frac{q^4 - q^2 + q}{4}$ . We consider  $\phi_{4,M}^+(1)$  in three parts. First of all,

$$\begin{aligned} m_4! \binom{N(4,q)}{m_4} &= \left(\frac{1}{4}(q^4 - q^2 + q)\right) \dots \left(\frac{1}{4}(q^4 - q^2 + q) - m_4 + 1\right) \\ &= \frac{1}{4^{m_4}} (q^{4m_4} - m_4 q^{4m_4-2} + O(q^{4m_4-3})) \\ &= \frac{q^{4m_4}}{4^{m_4}} (1 - m_4 q^{-2} + O(q^{-3})). \end{aligned} \quad (21)$$

Secondly,

$$\prod_j \frac{q^{-4jm_{4j}}}{m_{4j}!} = \frac{q^{-4 \sum j m_{4j}}}{\prod_j m_{4j}!}.$$

Finally  $\prod_j \frac{1}{(1 - q^{-4} + q^{-8})^{m_{4j}}} = \frac{1}{(1 - q^{-4} + q^{-8})^{m_4}} = 1 + O(q^{-4})$ . Multiplying together the three parts and collecting terms gives us  $\phi_{4,M}^+(1)$  equal to

$$\frac{q^{4m_4-4\sum j m_{4j}}}{4^{m_4} \prod_j m_{4j}!} (1 - m_4 q^{-2} + O(q^{-3})).$$

**Case:  $i \geq 5$**

From the definition of  $\phi_{i,M}^+(t)$  in Theorem 4.11,

$$\phi_{i,M}^+(1) = m_i! \binom{N(i,q)}{m_i} \prod_{j=1}^{\lfloor r/i \rfloor} \frac{q^{-ijm_{ij}}}{m_{ij}!(1 - q^{-i} + q^{-2i})^{m_{ij}}}.$$

Note that  $N(i,q) = N^+(i,q)$  for  $i \geq 2$ . We consider  $\phi_{i,M}^+(1)$  in three parts. First of all, since the exponent of the second term in  $N(i,q)$  is at least 3 less than the leading term, we have

$$m_i! \binom{N(i,q)}{m_i} = \frac{q^{im_i}}{i^{m_i}} (1 + O(q^{-3})). \quad (22)$$

Secondly,

$$\prod_j \frac{q^{-ijm_{ij}}}{m_{ij}!} = \frac{q^{-i\sum j m_{ij}}}{\prod_j m_{ij}!}.$$

Finally  $\prod_j \frac{1}{(1 - q^{-i} + q^{-2i})^{m_{ij}}} = \frac{1}{(1 - q^{-i} + q^{-2i})^{m_i}} = 1 + O(q^{-i})$ . Multiplying together the three parts and collecting terms gives us  $\phi_{i,M}^+(1)$  equal to

$$\frac{q^{im_i - i\sum j m_{ij}}}{i^{m_i} \prod_j m_{ij}!} (1 + O(q^{-3})).$$

□

**Corollary 4.16.** *For  $M = (m_{ij}) \in \mathcal{M}_{part}(r)$ , we get that*

$$\begin{aligned} \phi_{1,M}^+(1) &= \frac{1}{m_{11}!} \left( 1 + \left( -\frac{m_{11}^2}{2} + \frac{m_{11}}{2} \right) q^{-1} \right. \\ &\quad \left. + \left( \frac{m_{11}^4}{8} - \frac{5m_{11}^3}{12} - \frac{m_{11}^2}{8} - \frac{7m_{11}}{12} \right) q^{-2} + O(q^{-3}) \right), \\ \phi_{2,M}^+(1) &= \frac{1}{2^{m_{21}} m_{21}!} \left( 1 - m_{21} q^{-1} + \left( -\frac{m_{21}^2}{2} + \frac{3m_{21}}{2} \right) q^{-2} + O(q^{-3}) \right), \\ \phi_{3,M}^+(1) &= \frac{1}{3^{m_{31}} m_{31}!} (1 - m_{31} q^{-2} + O(q^{-3})), \\ \phi_{4,M}^+(1) &= \frac{1}{4^{m_{41}} m_{41}!} (1 - m_{41} q^{-2} + O(q^{-3})), \\ \phi_{i,M}^+(1) &= \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-3})) \text{ for } i \geq 5. \end{aligned}$$

*Proof.* Let  $(m_{ij}) \in \mathcal{M}_{part}(r)$ , that is  $(m_{ij})$  corresponds to a partition of  $r$ . Then for all  $i$  we have  $m_{ij} = 0$  for  $j \geq 2$ . Hence  $im_i - i \sum_j jm_{ij} = 0$ . Thus the first term of  $\phi_{i,M}^+(1)$  is a constant term for all  $i$  and the remaining terms are as in Lemma 4.15.  $\square$

We have expressions for the  $\phi_{i,M}^+(1)$  for different  $M \in \mathcal{M}(r)$ . The following lemma and corollary will help us understand more about them.

**Lemma 4.17.** *For a given  $i$ , the leading term of  $\phi_{i,M}^+(1)$  is a constant if and only if  $m_{ij} = 0$  for all  $j \geq 2$ .*

*Proof.* Let  $M = (m_{ij}) \in \mathcal{M}(r)$  such that for a given  $i$ ,  $m_{ij} = 0$  for all  $j \geq 2$ . By Lemma 4.15 the leading term of  $\phi_{i,M}^+(1)$  is  $q^{im_i - i \sum_j jm_{ij}}$ . Since  $m_{ij} = 0$  for  $j \geq 2$ , we get that  $im_i - i \sum_j jm_{ij} = 0$  and hence the leading term of  $\phi_{i,M}^+(1)$  is a constant.

Conversely suppose that the leading term of  $\phi_{i,M}^+(1)$  is a constant. Then by Lemma 4.15,  $i \sum_j m_{ij} - i \sum_j jm_{ij} = 0$ , that is,  $i \sum_j (1-j)m_{ij} = 0$ . Hence  $m_{ij} = 0$  for all  $j \geq 2$ .  $\square$

**Corollary 4.18.** *The leading term of  $\phi_{i,M}^+(1)$  is a constant for all  $i$  if and only if  $(m_{ij}) \in \mathcal{M}_{part}(r)$ .*

*Proof.* If the leading term of every  $\phi_{i,M}^+(1)$  is a constant then by Lemma 4.17,  $m_{ij} = 0$  for all  $j \geq 2$  and for all  $i$ . Hence  $(m_{ij})$  is a partition of  $r$ , that is  $(m_{ij}) \in \mathcal{M}_{part}(r)$ . Conversely, if  $(m_{ij})$  is a partition of  $r$ , then  $m_{ij} = 0$  for all  $i$  and  $j \geq 2$ . Then by Lemma 4.17 the leading term of  $\phi_{i,M}^+(1)$  is a constant for all  $i$ .  $\square$

The next two lemmas provide us with the tools which will enable us to sum the  $\phi_{i,M}^+(1)$  over all  $M \in \mathcal{M}(r)$ .

**Lemma 4.19.** *Suppose that for some  $k \in \mathbb{Z}^+$  and  $f_k : \mathbb{Z}^+ \rightarrow \mathbb{Q}$  we have, for each  $r$ ,*

$$a_r = \sum_{M \in \mathcal{M}_{part}(r)} f_k(m_{k1}) \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}.$$

*Then*

$$1 + \sum_{r=1}^{\infty} a_r t^r = \left( \sum_{m \geq 0} \frac{f_k(m) t^{mk}}{k^m m!} \right) \frac{e^{-\frac{t^k}{k}}}{1-t}.$$

*Proof.* For  $i \in \mathbb{Z}^+$ , consider the series

$$\frac{1}{1-t^i} = 1 + t^i + t^{2i} + \dots$$

The product of  $\frac{1}{1-t^i}$  taken over all  $i$  is equal to the generating function for the number of partitions (see [23]). That is, it is the series  $1 + \sum_{r=1}^{\infty} p_r t^r$  where  $p_r$



is the number of partitions of  $r$ . This is because each partition of  $r$  corresponds to a unique selection of terms  $t^{m_{i1}i}$  from the series above, one term for each  $i$ , such that  $\sum_i m_{i1}i = r$ . The corresponding partition has  $m_{i1}$  parts of size  $i$  for each  $i$ .

In the above scenario, each partition of  $r$  contributes  $t^r$  to the generating function. For the lemma we want the partition of  $r$  having  $m_{i1}$  parts of size  $i$ , for each  $i$ , to contribute  $f_k(m_{k1})t^r \prod_i \frac{1}{i^{m_{i1}m_{i1}!}}$  to our generating function. Hence each time we choose  $m_{i1}$  parts of size  $i$  for our partition of  $r$  we want it to contribute  $\frac{t^{im_{i1}}}{i^{m_{i1}m_{i1}!}}$  for  $i \neq k$  and  $\frac{f_k(m_{i1})t^{im_{i1}}}{i^{m_{i1}m_{i1}!}}$  when  $i = k$ .

For all  $i \neq k$ , consider the series

$$e^{\frac{t^i}{i}} = 1 + \frac{t^i}{i^1 1!} + \frac{t^{2i}}{i^2 2!} + \dots$$

and for  $i = k$  consider the series

$$\sum_{m \geq 0} \frac{f_k(m)t^{mk}}{k^m m!} = f_k(0) + \frac{f_k(1)t^k}{k^1 1!} + \frac{f_k(2)t^{2k}}{k^2 2!} + \dots$$

Then the partition  $(m_{ij}) \in \mathcal{M}_{part}(r)$  corresponds to selecting, for each  $i$ , the term involving  $t^{im_{i1}}$  from the  $i$ th series above, and its contribution to the generating function is  $f_k(m_{k1})t^r \prod_i \frac{1}{i^{m_{i1}m_{i1}!}}$ . Hence

$$\left( \sum_{m \geq 0} \frac{f_k(m)t^{mk}}{k^m m!} \right) \prod_{i \neq k} e^{\frac{t^i}{i}} = 1 + \sum_{r=1}^{\infty} a_r t^r$$

and this is the generating function we seek. Now

$$\prod_{i \neq k} e^{\frac{t^i}{i}} = e^{\frac{-t^k}{k}} \prod_i e^{\frac{t^i}{i}} = e^{\frac{-t^k}{k}} e^{\sum_i \frac{t^i}{i}} = e^{\frac{-t^k}{k}} e^{-\log(1-t)} = \frac{e^{\frac{-t^k}{k}}}{1-t}$$

and hence

$$1 + \sum_{r=1}^{\infty} a_r t^r = \left( \sum_{m \geq 0} \frac{f_k(m)t^{mk}}{i^m m!} \right) \frac{e^{\frac{-t^k}{k}}}{1-t}.$$

□

**Corollary 4.20.** *For each  $r > 0$ , let*

$$a_r = \sum_{M \in \mathcal{M}_{part}(r)} \prod_i \frac{1}{i^{m_{i1}m_{i1}!}}.$$

*Then  $1 + \sum_{r \geq 0} a_r t^r$  is equal to  $\frac{1}{1-t}$ . That is,  $a_r = 1$  for all  $r \geq 0$ .*

*Proof.* Letting  $f_k(m) = 1$  for all  $m$  in Lemma 4.19, we obtain

$$a_r = \sum_{M \in \mathcal{M}_{part}(r)} \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$$

and

$$\begin{aligned} 1 + \sum_{r=1}^{\infty} a_r t^r &= \left( \sum_{m \geq 0} \frac{t^{mk}}{k^m m!} \right) \frac{e^{-\frac{t^k}{k}}}{1-t} \\ &= \left( e^{\frac{t^k}{k}} \right) \frac{e^{-\frac{t^k}{k}}}{1-t} \\ &= \frac{1}{1-t}. \end{aligned}$$

Clearly all the coefficients in the expansion of  $\frac{1}{1-t}$  are 1. Hence  $a_r$  is equal to 1 for all  $r$ .  $\square$

**Lemma 4.21.** *Suppose that, for some  $k, \ell \in \mathbb{Z}^+$  and functions  $f_k, f_\ell : \mathbb{Z}^+ \rightarrow \mathbb{Q}$  and for each  $r$ ,*

$$a_r = \sum_{M \in \mathcal{M}_{part}(r)} f_k(m_{k1}) f_\ell(m_{\ell 1}) \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}.$$

*Then*

$$1 + \sum_{r=1}^{\infty} a_r t^r = \left( \sum_{m \geq 0} \frac{f_k(m) t^{mk}}{k^m m!} \right) \left( \sum_{m \geq 0} \frac{f_\ell(m) t^{m\ell}}{\ell^m m!} \right) \frac{e^{-\frac{t^k}{k}} e^{-\frac{t^\ell}{\ell}}}{1-t}.$$

*Proof.* As with the proof of Lemma 4.19, for all  $i \neq k$  and  $i \neq \ell$ , consider the series

$$e^{\frac{t^i}{i}} = 1 + \frac{t^i}{i^1 1!} + \frac{t^{2i}}{i^2 2!} + \dots,$$

for  $i = k$  consider the series

$$\sum_{m \geq 0} \frac{f_k(m) t^{mk}}{k^m m!} = f_k(0) + \frac{f_k(1) t^k}{k^1 1!} + \frac{f_k(2) t^{2k}}{k^2 2!} + \dots$$

and for  $i = \ell$  consider the series

$$\sum_{m \geq 0} \frac{f_\ell(m) t^{m\ell}}{\ell^m m!} = f_\ell(0) + \frac{f_\ell(1) t^\ell}{\ell^1 1!} + \frac{f_\ell(2) t^{2\ell}}{\ell^2 2!} + \dots$$

Each partition  $(m_{ij}) \in \mathcal{M}_{part}(r)$ , corresponds to a product of a unique selection of terms from the series above, one for each  $i$ , and the contribution to the generating function is  $f_k(m_{k1}) f_\ell(m_{\ell 1}) t^r \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$ . Hence

$$1 + \sum_{r=1}^{\infty} a_r t^r = \left( \sum_{m \geq 0} \frac{t^{mk} f_k(m)}{k^m m!} \right) \left( \sum_{m \geq 0} \frac{t^{m\ell} f_\ell(m)}{\ell^m m!} \right) \prod_{i \neq k, \ell} e^{\frac{t^i}{i}}$$

and hence is the generating function we require. We see now that

$$\prod_{i \neq k, \ell} e^{\frac{t^i}{i}} = e^{\frac{-t^k}{k}} e^{\frac{-t^\ell}{\ell}} \prod_i e^{\frac{t^i}{i}} = e^{\frac{-t^k}{k}} e^{\frac{-t^\ell}{\ell}} e^{\sum \frac{t^i}{i}} = e^{\frac{-t^k}{k}} e^{\frac{-t^\ell}{\ell}} e^{-\log(1-t)} = \frac{e^{\frac{-t^k}{k}} e^{\frac{-t^\ell}{\ell}}}{1-t}.$$

Hence the generating function we require is

$$\left( \sum_{m \geq 0} \frac{f_k(m) t^{mk}}{k^m m!} \right) \left( \sum_{m \geq 0} \frac{f_\ell(m) t^{m\ell}}{\ell^m m!} \right) \frac{e^{\frac{-t^k}{k}} e^{\frac{-t^\ell}{\ell}}}{1-t}.$$

□

We now have the techniques required to determine  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$ . The following theorem does this.

**Theorem 4.22.** *For  $r \in \mathbb{Z}^+$ ,  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n) = 1 - q^{-2} + O(q^{-3})$ .*

*Proof.* The proof will consist of three parts. The first part will calculate the constant term in the expansion of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$ , the second part will calculate the coefficient of  $q^{-1}$  and the third part will calculate the coefficient of  $q^{-2}$ .

Before starting the first part, we note that the expansion of  $\frac{1-q^{-5}}{1+q^{-3}}$  is

$$1 - q^{-3} - q^{-5} + O(q^{-6})$$

and by Theorem 4.14 we multiply this with  $\sum_{M \in \mathcal{M}(r)} \prod_i \phi_{i,M}^+(1)$  to produce  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$ . Thus the constant term, the  $q^{-1}$  term and the  $q^{-2}$  term in  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  are the same as those in the expansion of  $\sum_{M \in \mathcal{M}(r)} \prod_i \phi_{i,M}^+(1)$ .

#### Constant Term:

By Corollary 4.18, the leading term of  $\phi_{i,M}^+(1)$  is a constant if and only if  $M \in \mathcal{M}_{\text{part}}(r)$ . From Corollary 4.16, we can see that this leading term is  $\frac{1}{i^{m_{i1}} m_{i1}!}$  if  $M \in \mathcal{M}_{\text{part}}(r)$  so the constant term in  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  is

$$\sum_{M \in \mathcal{M}_{\text{part}}(r)} \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}.$$

By Corollary 4.20 this equals 1 for all  $r$ . Hence 1 is the constant term of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  for all  $r$ .

#### Coefficient of $q^{-1}$

For the coefficient of  $q^{-1}$ , we will first sum over all  $M \in \mathcal{M}_{\text{part}}(r)$  and then look at those  $M \notin \mathcal{M}_{\text{part}}(r)$  that also contribute to the  $q^{-1}$  term.

Let  $M = (m_{ij}) \in \mathcal{M}_{part}(r)$ . By Corollary 4.16 there are exactly two ways to produce a  $q^{-1}$  term. The first is to multiply the  $q^{-1}$  term in  $\phi_{1,M}^+(1)$  with the constant term from each of the remaining  $\phi_{i,M}^+(1)$ . The second way is to multiply the  $q^{-1}$  term in  $\phi_{2,M}^+(1)$  with the constant term in each of the remaining  $\phi_{i,M}^+(1)$ .

The first way produces  $\left(-\frac{m_{11}^2}{2} + \frac{m_{11}}{2}\right) \prod_i \frac{1}{i^{m_{i1}} m_{i1}!} q^{-1}$ . Summing this over all  $M \in \mathcal{M}_{part}(r)$ , gives the generating function as

$$-\frac{1}{2} \left( \sum_{m \geq 0} \frac{m^2 t^m}{m!} \right) \frac{e^{-t}}{1-t} + \frac{1}{2} \left( \sum_{m \geq 0} \frac{m t^m}{m!} \right) \frac{e^{-t}}{1-t}$$

by Lemma 4.19, using  $f_1(m) = m^2$  in the first case and  $f_1(m) = m$  in the second case. By Lemma 4.2, we get the generating function for the coefficient of  $q^{-1}$  arising in this way to be

$$\left( -\frac{1}{2}(t^2 + t)e^t + \frac{1}{2}te^t \right) \frac{e^{-t}}{1-t} = \left( \frac{-t^2}{2} \right) \frac{1}{1-t}.$$

The second way an  $M \in \mathcal{M}_{part}(r)$  can produce a  $q^{-1}$  term, produces  $-m_{21} \prod_i \frac{1}{i^{m_{i1}} m_{i1}!} q^{-1}$ . Summing this over all  $M \in \mathcal{M}_{part}(r)$ , gives the generating function for the contribution to the coefficient of  $q^{-1}$  as

$$\left( - \sum_{m \geq 0} \frac{m t^{2m}}{2^m m!} \right) \frac{e^{-\frac{t^2}{2}}}{1-t}$$

by Lemma 4.19, using  $f_2(m) = -m$ . By Lemma 4.2 with  $b = 1$  and  $k = 2$ , we get  $\sum_{m \geq 0} \frac{m t^{2m}}{2^m m!} = \frac{t^2}{2} e^{\frac{t^2}{2}}$ , so the generating function for the coefficient of  $q^{-1}$  arising in this way is

$$\left( \frac{-t^2}{2} e^{\frac{t^2}{2}} \right) \frac{e^{-\frac{t^2}{2}}}{1-t} = \left( \frac{-t^2}{2} \right) \frac{1}{1-t}.$$

Summing together these two functions gives the generating function for the contribution to the coefficient of  $q^{-1}$  from  $M \in \mathcal{M}_{part}(r)$ , namely  $\frac{-t^2}{1-t}$ .

We need to look at  $M = (m_{ij}) \notin \mathcal{M}_{part}(r)$  that also contribute to the  $q^{-1}$  term. By Lemma 4.17 there exists an  $i$  such that the leading term of  $\phi_{i,M}^+(1)$  is not a constant. Hence for  $M$  to contribute to the  $q^{-1}$  term there is a unique  $i$  such that the leading term of  $\phi_{i,M}^+(1)$  is not a constant and this leading term is  $q^{-1}$ . Thus  $i \sum_j m_{ij} - i \sum_j j m_{ij} = -1$ . Rewriting the equation we see that we need  $\sum_j (1-j)m_{ij} = \frac{-1}{i}$ . Since the left hand side is an integer, this equality can only hold for  $i = 1$ . Thus we need  $\phi_{i,M}^+(1)$  for  $i \geq 2$  to have constant leading term and  $\sum_j (1-j)m_{1j} = -1$ . By Lemma 4.17 The first condition implies that  $m_{ij} = 0$  for  $i \geq 2$  and  $j \geq 2$  while the second condition implies that  $m_{12} = 1$

and  $m_{1j} = 0$  for  $j \geq 3$ . Hence if  $(m_{ij}) \notin \mathcal{M}_{part}(r)$  and contributes to the  $q^{-1}$  term then

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{12} = 1 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2). \end{cases} \quad (23)$$

Note that there are no such  $(m_{ij})$  when  $r = 1$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r-2)$ . Moreover, each element of  $\mathcal{M}_{part}(r-2)$  occurs in this way so we think of an  $(m_{ij})$  as in (23) as a partition of  $r-2$  with  $m_{12} = 1$  appended to it.

We can see from Lemma 4.15, that for each  $M = (m_{ij})$  as in (23), for  $i \geq 2$ ,  $\phi_{i,M}^+(1)$  has constant term  $\frac{1}{i^{m_{i1}} m_{i1}!}$  and for  $i = 1$  the coefficient of  $q^{-1}$  is  $\frac{1}{m_{11}!}$ . Hence each of these  $(m_{ij})$  produces  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  as the coefficient of  $q^{-1}$ . Summing this over all  $M \notin \mathcal{M}_{part}(r)$  that contribute to the  $q^{-1}$  term is then the same as summing over all  $M' \in \mathcal{M}_{part}(r-2)$ . By Lemma 4.19 the generating function for the sum of  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  over all partitions of  $r$  is  $\frac{1}{1-t}$ . The generating function for the sum of  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  over all partitions of  $r-2$  has no terms of degree less than 2 and so is  $\frac{t^2}{1-t}$ . Hence the generating function for the contribution to the coefficient of  $q^{-1}$  from  $M \notin \mathcal{M}_{part}(r)$  is  $\frac{t^2}{1-t}$ .

Adding the generating functions for contributions due to all  $M \in \mathcal{M}_{part}(r)$  and  $M \notin \mathcal{M}_{part}(r)$  to the coefficient of  $q^{-1}$  gives us 0. Hence for all  $r$  there is no  $q^{-1}$  term in the expansion of  $\lim_{n \rightarrow \infty} c_{GL,r}(n)$ .

### Coefficient of $q^{-2}$

We take the same approach in calculating the coefficient of  $q^{-2}$ . We first sum over all  $M \in \mathcal{M}_{part}(r)$ . Let  $M \in \mathcal{M}_{part}(r)$ . From the expansion of the  $\phi_{i,M}^+(1)$  in Corollary 4.16 we see that there are several ways to form a  $q^{-2}$  term. Each of these ways corresponds to a line of Table 1. For each line in Table 1, the contribution from  $\phi_{i,M}^+$  for  $i \geq 5$  is 1. If a line contains exactly one non-1 entry then that entry is a certain function of  $m_{k1}$  for some  $k$ , say  $f_k(m_{k1})$  and this line corresponds to a summand  $a_r$  of the coefficient of  $q^{-2}$ , where  $a_r$  is as given in Lemma 4.19. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemma 4.19 and Lemma 4.2 and recorded in the last entry for this line. There is one further line that contains two non-1 entries which are certain functions  $f_k(m_{k1})$  with  $k = 1$  and  $f_\ell(m_{\ell 1})$  with  $\ell = 2$  and this line corresponds to a summand  $a_r$  of the coefficient of  $q^{-1}$ , where  $a_r$  is as given in Lemma 4.21. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemmas 4.21 and Lemma 4.2 and recorded in the last entry for this line.

The sum of all the generating functions is the generating function for the coefficient of  $q^{-2}$  due to  $M \in \mathcal{M}_{part}(r)$ . It is  $\frac{-t}{1-t}$ .

We now look for the contribution to the  $q^{-2}$  term in  $\prod_i \phi_{i,M}^+(1)$  from  $M \notin$

$\phi_{1,M}^+(1)$	$\phi_{2,M}^+(1)$	$\phi_{3,M}^+(1)$	$\phi_{4,M}^+(1)$	Generating Function
1	1	1	$-m_{41}q^{-2}$	$\left(-\frac{t^4}{4}\right) \frac{1}{1-t}$
1	1	$-m_{31}q^{-2}$	1	$\left(-\frac{t^3}{3}\right) \frac{1}{1-t}$
1	$\frac{3m_{21}}{2}q^{-2}$	1	1	$\left(\frac{3t^2}{4}\right) \frac{1}{1-t}$
1	$-\frac{m_{21}^2}{2}q^{-2}$	1	1	$\left(-\frac{t^4}{8} - \frac{t^2}{4}\right) \frac{1}{1-t}$
$-\frac{m_{11}(m_{11}-1)}{2}q^{-1}$	$-m_{21}q^{-1}$	1	1	$\left(\frac{t^4}{4}\right) \frac{1}{1-t}$
$-\frac{7m_{11}}{12}q^{-2}$	1	1	1	$\left(-\frac{7t}{12}\right) \frac{1}{1-t}$
$-\frac{m_{11}^2}{8}q^{-2}$	1	1	1	$\left(\frac{-t^2-t}{8}\right) \frac{1}{1-t}$
$-\frac{5m_{11}^3}{12}q^{-2}$	1	1	1	$\left(\frac{-5(t^3+3t^2+t)}{12}\right) \frac{1}{1-t}$
$\frac{m_{11}^4}{8}q^{-2}$	1	1	1	$\left(\frac{t^4+6t^3+7t^2+t}{8}\right) \frac{1}{1-t}$
Total				$(-t) \frac{1}{1-t}$

Table 1: Producing the generating function for the coefficient of  $q^{-2}$  in the expansion of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  due to  $M \in \mathcal{M}_{\text{part}}(r)$ .

$\mathcal{M}_{\text{part}}(r)$ . For each  $i$  we need to choose a term involving  $q^{a_i}$  from  $\phi_{i,M}^+(1)$  such that  $\sum_i a_i = -2$ . In particular each  $a_i$  satisfies  $0 \geq a_i \geq -2$ . We will use Lemma 4.15.

For  $i \geq 3$  the exponent of  $q$  either is  $i \sum_j (1-j)m_{ij}$  or is at most  $i \sum_j (1-j)m_{ij} - 3$ . The only possible value out of  $0, -1, -2$  is 0 and hence for each such  $M$  we must have  $m_{ij} = 0$  for all  $i \geq 3, j \geq 2$  and  $a_i = 0$  for all  $i \geq 3$ .

Since we require  $M \notin \mathcal{M}_{\text{part}}(r)$  there must exist an  $i_0$  equal to 1 or 2 and  $j_0 \geq 2$  such that  $m_{i_0 j_0} > 0$ . By Lemma 4.15,

$$-2 \leq a_{i_0} \leq i_0 \sum_j (1-j)m_{i_0 j} \leq i_0(1-j_0)m_{i_0 j_0}. \quad (24)$$

Hence either

- $i_0 = 2$  and  $(j_0, m_{2j_0}) = (2, 1)$  and  $m_{2j} = 0$  for all  $j \geq 3$ ; or
- $i_0 = 1$  and  $(j_0, m_{1j_0}) = (2, 1), (2, 2)$  or  $(3, 1)$  with  $m_{1j} = 0$  for all  $j \neq 1, j_0$ .

We consider each of these four cases in turn.

Let us consider the first scenario where  $i_0 = 2, j_0 = 2, m_{22} = 1$  and  $m_{2j} = 0$  for  $j \geq 3$ . It follows from (24) that  $a_2 = -2$  and hence  $a_i = 0$  for  $i \neq 2$ . Thus the first type of  $(m_{ij}) \notin \mathcal{M}_{\text{part}}(r)$  that contributes to the  $q^{-2}$  term is as follows

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{22} = 1 \\ m_{2j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \neq 2, j \geq 2). \end{cases} \quad (25)$$

Note that there are no such  $(m_{ij})$  for  $r < 4$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (2, 2)$  and  $m'_{22} = 0$  then  $M' \in \mathcal{M}_{part}(r - 4)$ . So we think of an  $(m_{ij})$  as in (25) as a partition of  $r - 4$  with  $m_{22} = 1$  appended to it.

For  $(m_{ij})$  as in (25), we have  $m_2 = m_{21} + 1$  and  $m_i = m_{i1}$  for  $i \neq 2$ . By Lemma 4.15 we obtain the  $\phi_{i,M}^+(1)$  as follows:

$$\begin{aligned}\phi_{1,M}^+(1) &= \frac{1}{m_{11}!}(1 + O(q^{-1})) \\ \phi_{2,M}^+(1) &= \frac{q^{-2}}{2^{m_{21}}m_{21}!}\left(\frac{1}{2}\right)(1 + O(q^{-1}))\end{aligned}$$

and for  $i \geq 3$

$$\phi_{i,M}^+(1) = \frac{1}{i^{m_{i1}}m_{i1}!}(1 + O(q^{-1})).$$

The ' $\frac{1}{2}$ ' inside the  $\phi_{2,M}^+(1)$  occurs because  $2^{m_2} = 2^{m_{21}}2^1$ .

There is only one way to make up a  $q^{-2}$  term and the first line in Table 2 corresponds to forming a  $q^{-2}$  term in this way.

From now on we will let  $i_0 = 1$  so  $m_{2j} = 0$  for all  $i \geq 2, j \geq 2$ . If  $(j_0, m_{1j_0}) = (3, 1)$  or  $(2, 2)$  then the leading term of  $\phi_{1,M}^+(1)$  involves  $q^{-2}$  so  $a_1 = -2$  and  $a_i = 0$  for all  $i \geq 2$ . Hence we get two more types of nonpartitions which contribute to the  $q^{-2}$  term, namely

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{13} = 1 \\ m_{1j} = 0 \ (j \neq 1, 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2) \end{cases} \quad (26)$$

and

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{12} = 2 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2). \end{cases} \quad (27)$$

Note that there are no such  $(m_{ij})$  as in (26) for  $r < 3$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 3)$  and  $m'_{13} = 0$  then  $M' \in \mathcal{M}_{part}(r - 3)$ . So we think of an  $(m_{ij})$  as in (26) as a partition of  $r - 3$  with  $m_{13} = 1$  appended to it.

Also note that there are no such  $(m_{ij})$  as in (27) for  $r < 4$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r - 4)$ . So we think of an  $(m_{ij})$  as in (27) as a partition of  $r - 4$  with  $m_{12} = 2$  appended to it.

For  $(m_{ij})$  as in (26), we have  $m_1 = m_{11} + 1$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we produce  $\phi_{i,M}^+(1)$  as

$$\phi_{1,M}^+(1) = \frac{q^{-2}}{m_{11}!} (1 + O(q^{-1}))$$

and for  $i \geq 2$

$$\phi_{i,M}^+(1) = \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-1})).$$

There is only one way to make up a  $q^{-2}$  term and the second line in Table 2 corresponds to forming a  $q^{-2}$  term in this way.

For  $(m_{ij})$  as in (27), we have  $m_1 = m_{11} + 2$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we obtain the  $\phi_{i,M}^+(1)$  as follows:

$$\phi_{1,M}^+(1) = \frac{q^{-2}}{m_{11}!2!} (1 + O(q^{-1}))$$

and for  $i \geq 2$

$$\phi_{i,M}^+(1) = \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-1})).$$

There is again only one way to make a  $q^{-2}$  term here and the third line in Table 2 corresponds to forming a  $q^{-2}$  term in this way. Note that  $\prod_j m_{1j}! = m_{1j}!2!$ , so we can take the half out as a factor.

Finally  $i_0 = 1$ ,  $j_0 = 2$  and  $m_{12} = 1$  with  $m_{1j} = 0$  for  $j \geq 3$  provides us with our last case. Here we have

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{12} = 1 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2) \end{cases} \quad (28)$$

Note that there are no such  $(m_{ij})$  for  $r < 2$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r-2)$ . So we think of an  $(m_{ij})$  as in 28 as a partition of  $r-2$  with  $m_{12} = 1$  appended to it.

For  $(m_{ij})$  as in (28), we have  $m_1 = m_{11} + 1$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we obtain the  $\phi_{i,M}^+(1)$  as follows

$$\phi_{1,M}^+(1) = \frac{q^{-1}}{m_{11}!} \left( 1 - \left( \frac{m_{11}^2}{2} + \frac{m_{11}}{2} \right) q^{-1} + O(q^{-2}) \right)$$

$$\phi_{2,M}^+(1) = \frac{1}{2^{m_{21}} m_{21}!} (1 - m_{21} q^{-1} + O(q^{-2}))$$

and for  $i \geq 3$

$$\phi_{i,M}^+(1) = \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-1})).$$



$M$ as in:	$\phi_{1,M}^+(1)$	$\phi_{2,M}^+(1)$	$\phi_{M,3+}^+(1)$	Generating Function
(25)	1	$\frac{1}{2}$	1	$\left(\frac{t^4}{2}\right) \frac{1}{1-t}$
(26)	1	1	1	$\left(t^3\right) \frac{1}{1-t}$
(27)	$\frac{1}{2}$	1	1	$\left(\frac{t^4}{2}\right) \frac{1}{1-t}$
(28)	1	$-m_{21}$	1	$\left(\frac{-t^4}{2}\right) \frac{1}{1-t}$
(28)	$-\frac{m_{11}^2}{2}$	1	1	$\left(-\frac{t^3}{2} - \frac{t^4}{2}\right) \frac{1}{1-t}$
(28)	$-\frac{m_{11}}{2}$	1	1	$\left(-\frac{t^3}{2}\right) \frac{1}{1-t}$
	Total			0

Table 2: Producing the generating function for the coefficient of  $q^{-2}$  in the expansion of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  due to  $M \in \mathcal{M}(r)$  corresponding to nonpartitions.

We have now that  $\phi_{1,M}^+(1)$  has leading term involving  $q^{-1}$  so we can choose  $a_1 = -1$  and  $a_2 = -1$ , or  $a_1 = -2$  and  $a_2 = 0$ . The former corresponds to line four of Table 2 while the latter corresponds to lines five and six of that table.

The first column of Table 2 indicates the type of array  $M$  used to produce the  $\phi_{i,M}^+(1)$ . If a line contains exactly one nonconstant entry then that entry is a certain function of  $m_{k1}$  for some  $k$ , say  $f_k(m_{k1})$  and this line corresponds to a summand  $a_r$  of the coefficient of  $q^{-2}$ , where  $a_r$  is as given in Lemma 4.19. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemmas 4.19 and Lemma 4.2 and recorded in the last entry for this line. There are lines in which all the entries are constant and we produce the generating functions by Corollary 4.20 in these cases. Each generating function has been multiplied by  $t^b$  for the positive integer  $b$  such that the  $(m_{ij})$ 's in question correspond to partitions of  $r - b$ .

Adding all the generating functions together for those  $M \notin \mathcal{M}_{\text{part}}(r)$  gives 0. So the generating function for the coefficient of  $q^{-2}$  is just what is produced by the  $M \in \mathcal{M}_{\text{part}}(r)$  which is  $\frac{-t}{1-t}$ . In the expansion of this, the coefficient of  $t^r$  is  $-1$  for all  $r \geq 1$ . Hence for all  $r$ , the coefficient of  $q^{-2}$  is  $-1$ . This completes the proof that for all  $r \in \mathbb{Z}^+$ , the  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n) = 1 - q^{-2} + O(q^{-3})$ .  $\square$

Theorem 1.1 for general linear groups now follows from Theorem 4.14 and Theorem 4.22.

It is a basic assumption of Section 4 that  $r$ , the dimension of the invariant subspace, is greater than or equal to 1. However, we could have relaxed that assumption, allowed  $r$  to equal zero and worked through the section in the same manner. If we had allowed  $r$  to equal zero then Theorem 4.11 would state that  $C_{\text{GL},r}(t) = C_{\text{GL}}(t)$  when  $r = 0$ , since the only matrix in  $\mathcal{M}(0)$  is the matrix containing only zeroes. In the proof of Theorem 4.22, the generating function for the coefficient of  $q^{-2}$  in  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  is  $\frac{-t}{1-t}$ . So for  $r = 0$ , the resulting answer for the limit of  $c_{\text{GL},0}(n)$  would be  $1 - q^{-3} + O(q^{-4})$ , since the coefficient of  $t^0$  in  $\frac{-t}{1-t}$  (which gives the coefficient of  $q^{-2}$  when  $r = 0$ ) is 0.

These are exactly the answers that should arise, because when  $r = 0$  the invariant subspace has dimension 0 and hence our matrix group  $\text{GL}(V)_U$  is equal to  $\text{GL}(V)$ . By Theorem 3.4 the limiting proportion of cyclic matrices in  $\text{GL}(V)$  is indeed  $1 - q^{-3} + O(q^{-4})$ .

### 4.3 Inside the Matrix Algebras

We move to looking for cyclic matrices inside the matrix algebra of all matrices which fix a subspace  $U$  of the vector space  $V$ . We now consider the subalgebra  $\text{M}(V)_U$ , that is, we include matrices whose characteristic polynomial has zero constant term, into our calculations.

Let  $\Gamma_{\text{M},r}(n)$  be the set of all cyclic matrices in  $\text{M}(V)_U$  where  $n$ , the dimension of the vector space  $V$ , will vary and  $r$ , the dimension of the subspace  $U$ , will remain fixed. Then by Lemma 2.9 there is a one-to-one correspondence between the set of orbits of  $\text{GL}(V)_U$  in its action on  $\Gamma_{\text{M},r}(n)$  by conjugation and the set of pairs of monic polynomials  $(f, h)$  over  $\mathbb{F}_q$  where  $f$  is of degree  $r$ ,  $h$  is of degree  $n$  and  $f$  divides  $h$ . We denote by  $\Gamma_{f,h}$  the orbit of  $\Gamma_{\text{M}}(n)$  containing those matrices with minimal polynomial  $f$  on  $U$  and minimal polynomial  $h$  on  $V$ . Note that when  $h$  has nonzero constant term, this  $\Gamma_{f,h}$  is the same as the set  $\Gamma_{f,h}$  defined in Section 4.1. Thus, by Equation (12) we have

$$|\Gamma_{f,h}| = \frac{|\text{GL}(V)_U|}{\text{Cent}(h)}.$$

We digress briefly to define  $\omega_r(n)$  and prove a result about it.

**Definition 4.23.** Let  $\omega_r(n) = \frac{|\text{GL}(V)_U|}{|\text{M}(V)_U|}$  where  $V$  has dimension  $n$  and  $U$  has dimension  $r$ .

**Lemma 4.24.** For some  $(n - r)$ -dimensional subspace  $W$ ,

$$\omega_r(n) = \frac{|\text{GL}(V)_U|}{|\text{M}(V)_U|} = \frac{|\text{GL}(V)_{U \oplus W}|}{|\text{M}(V)_{U \oplus W}|} = \prod_{i=1}^r (1 - q^{-i}) \prod_{i=1}^{n-r} (1 - q^{-i}).$$

Also

$$\lim_{n \rightarrow \infty} \omega_r(n) = \begin{cases} 1 - 2q^{-1} + q^{-3} + O(q^{-5}), & \text{if } r = 1 \\ 1 - 2q^{-1} - q^{-2} + 3q^{-3} + O(q^{-5}), & \text{if } r = 2 \\ 1 - 2q^{-1} - q^{-2} + 2q^{-3} + 2q^{-4} + O(q^{-5}), & \text{if } r = 3 \\ 1 - 2q^{-1} - q^{-2} + 2q^{-3} + q^{-4} + O(q^{-5}), & \text{if } r \geq 4 \end{cases}$$

*Proof.* This is proved, for example, in [4, Lemma 4.3.2].  $\square$

Recall that  $\mathcal{P}$  is the set of all monic polynomials over  $\mathbb{F}_q$  and that  $\mathcal{P}_i$  is the subset of  $\mathcal{P}$  that contains the constant polynomial 1 and those polynomials whose irreducible factors all have degree  $i$ . Also recall that  $\alpha(h; r)$  denotes the number of distinct degree  $r$  factors of the polynomial  $h$ .

Now denote by  $c_{M,r}(n)$  the proportion of cyclic matrices in  $M(V)_U$ . Then it follows that

$$\begin{aligned}
c_{M,r}(n) &= \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n \\ \deg(f|h)=r}} \frac{|\Gamma_{f,h}|}{|M(V)_U|} \\
&= \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{\alpha(h;r) |\Gamma_{f,h}| |\mathrm{GL}(V)_U|}{|M(V)_U| |\mathrm{GL}(V)_U|} \\
&= \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{\alpha(h;r) \omega_r(n)}{Cent(h)}.
\end{aligned} \tag{29}$$

We have that

$$\frac{c_{M,r}(n)}{\omega_r(n)} = \sum_{\substack{h \in \mathcal{P} \\ \deg(h)=n}} \frac{\alpha(h;r)}{Cent(h)}$$

which we will calculate by a similar method to that used for calculating  $c_{GL,r}(n)$  in Section 4.1. Let

$$C_{M,r}(t) = \sum_{n=r}^{\infty} \left( \frac{c_{M,r}(n)}{\omega_r(n)} \right) t^n$$

be the 'weighted' generating function for the proportion of cyclic matrices in  $M(V)_U$ . Note that the coefficients of  $C_{M,r}(t)$  are not the proportions we desire, as they are 'weighted' by  $\omega_r(n)$ . Note also that the sum starts from  $n = r$  because  $r \leq \dim(V)$ .

Recall that Equation 16 gave us a formal power series for each irreducible polynomial and that we collected all irreducible polynomials of degree  $i$  to form

$$\Phi_i^+ = \left( 1 + \sum_{j=1}^{\infty} \frac{s_{ij} t^{ij}}{Cent(i,j)} \right)^{N^+(i,q)}.$$

We will slightly modify these  $\Phi_i^+$  by making just one small change and we will call the new power series  $\Phi_i$ . Because we want to include the irreducible polynomial  $t$  in our calculations now, we make the exponent equal to  $N(i,q)$  instead of  $N^+(i,q)$ . Since  $N(i,q) = N^+(i,q)$  for  $i \geq 2$  it follows that  $\Phi_i = \Phi_i^+$  for  $i \geq 2$ . So we define

$$\Phi_i = \left( 1 + \sum_{j=1}^{\infty} \frac{s_{ij} t^{ij}}{Cent(i,j)} \right)^{N(i,q)}.$$

Now we give the first of several corollaries. The first follows immediately from Lemma 4.7, the only difference being that the summands now include polynomials with zero constant term.

**Corollary 4.25.** *As a power series,  $\Phi_i$  satisfies*

$$(1). \Phi_i = \sum_{h_i \in \mathcal{P}_i} \frac{\left( \prod_j s_{ij}^{\tau(h_i; i, j)} \right) t^{\deg(h_i)}}{\text{Cent}(h_i)},$$

$$(2). \prod_{i=1}^{\infty} \Phi_i = \sum_{h \in \mathcal{P}} \frac{\left( \prod_j s_{ij}^{\tau(h; i, j)} \right) t^{\deg(h)}}{\text{Cent}(h)}.$$

Recall that in Section 3.1 we defined  $\alpha(h; r)$  by

$$\alpha(h; r) = \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \alpha(h_i; r_i, M_i)$$

where  $h_i$  was the product of all the irreducible degree  $i$  factors of  $h$ , we assigned  $r_i = i \sum_j j m_{ij}$ ,  $M_i = M_i(M)$  is as in (14) and the overall term  $\alpha(h_i; r_i, M_i)$  referred to the number of degree  $r_i$  factors of  $h_i$  that corresponded with the array  $M$ .

Similarly to Definition 4.8 we define  $\Phi_{i, M, \alpha}$ , for each  $i$ , by

$$\Phi_{i, M, \alpha} = \Phi_{i, M, \alpha}(t) := \sum_{h_i \in \mathcal{P}_i} \alpha(h_i; r_i, M_i) \frac{t^{\deg(h_i)}}{\text{Cent}(h_i)}. \quad (30)$$

The next corollary, which follows immediately from Lemma 4.9, tells us how to obtain  $C_{M, r}(t)$  using the  $\Phi_{i, M, \alpha}$ .

**Corollary 4.26.**

$$C_{M, r}(t) = \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^{\infty} \Phi_{i, M, \alpha}(t)$$

When given  $\Phi_i$  as input, the procedure PHIALPHA from Section 4.1 produces  $\Phi_{i, M, \alpha}(t)$ . So we have the following theorem which is similar to Theorem 4.11. The proof is the same as that of Theorem 4.11 with  $N(i, q)$  substituted for  $N^+(i, q)$ .

**Theorem 4.27.** *Let  $\phi_{i, M}(t) = m_i! \binom{N(i, q)}{m_i} \prod_j \frac{(tq^{-1})^{ij m_{ij}}}{m_{ij}! (1 - q^{-i} + t^i q^{-2i})^{m_{ij}}}$ . Then*

$$C_{M, r}(t) = C_M(t) \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i, M}(t)$$

Moreover,  $\phi_{i, M}(t) = 1$  for all  $M \in \mathcal{M}(r)$  when  $i > r$ .

Theorem 4.27 gives us a direct formula for calculating the generating function  $C_{M, r}(t)$  for any  $r$ . Now we want to know the limit of the coefficients as  $n$  tends to infinity. We first need to know the convergence properties of  $C_{M, r}(t)$ . The following corollary follows immediately from Lemma 4.12.

**Corollary 4.28.** *Let  $M \in \mathcal{M}(r)$ . Then for any  $i \leq r$  the power series expansion of*

$$\phi_{i,M}(t) = m_i! \binom{N(i,q)}{m_i} \prod_j \frac{(tq^{-1})^{ijm_{ij}}}{m_{ij}!(1 - q^{-i} + t^i q^{-2i})^{m_{ij}}}$$

*is convergent for  $|t| < q(q^i - 1)^{1/i}$ .*

This again follows since the only difference between  $\phi_{i,M}$  and  $\phi_{i,M}^+$  is when  $i = 1$  where we have  $N(1, q) = N^+(1, q) + 1$ . Similarly we have the next corollary.

**Corollary 4.29.** *The power series expansion of  $\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(t)$  is convergent for  $|t| < q(q - 1)$ .*

Now that the convergence properties are established we produce the first main result of the section - a theorem giving a formula for the limit of the coefficients of  $t^n$  in  $C_{M,r}(t)$  as  $n$  tends to infinity.

**Theorem 4.30.** *For  $r \in \mathbb{Z}^+$*

$$\lim_{n \rightarrow \infty} \left( \frac{c_{M,r}(n)}{\omega_r(n)} \right) = \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1).$$

*Moreover,  $|\frac{c_{M,r}(n)}{\omega_r(n)} - \lim_{n \rightarrow \infty} \frac{c_{M,r}(n)}{\omega_r(n)}| = O(d^{-n})$  for any  $d$  such that  $1 < d < q(q - 1)$ .*

*Proof.* By [22],  $(1 - t)C_M(t)$  is convergent for  $|t| < q^2$ , and by Corollary 4.29,  $\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(t)$  is convergent for  $|t| < q(q - 1)$ . Hence  $C_{M,r}(t)$  is convergent for  $|t| < q(q - 1)$ .

By Lemma 3.3,  $(1 - t)C_{M,r}(t)$  evaluated at  $t = 1$  gives the limit of the coefficients of  $C_{M,r}(t)$ , that is  $\lim_{n \rightarrow \infty} \left( \frac{c_{M,r}(n)}{\omega_r(n)} \right)$ . But

$$C_{M,r}(t) = C_M(t) \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(t)$$

by Theorem 4.27 and we know already that  $(1 - t)C_M(t)$  evaluated at  $t = 1$  is  $\frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})}$  by Theorem 3.7. Hence

$$\lim_{n \rightarrow \infty} \left( \frac{c_{M,r}(n)}{\omega_r(n)} \right) = \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1).$$

We showed above that  $(1 - t)C_{M,r}(t)$  is convergent for  $|t| < q(q - 1)$ , so the final assertion about the rate of convergence follows by Lemma 3.3.  $\square$

We now look at the power series expansion of  $\phi_{i,M}(1)$  for all  $i$ . This will help us obtain a power series expansion of  $\lim_{n \rightarrow \infty} c_{M,r}(t)$ .

**Lemma 4.31.** *Let  $M \in \mathcal{M}(r)$  and set  $m_i = \sum_j m_{ij}$ . Then with the notation of Theorem 4.27, we have*

$$\begin{aligned}
\phi_{1,M}(1) &= \frac{q^{m_1 - \sum_j j m_{1j}}}{\prod_j m_{1j}!} \left( 1 + \left( \frac{-m_1^2}{2} + \frac{3m_1}{2} \right) q^{-1} \right. \\
&\quad \left. + \left( \frac{m_1^4}{8} - \frac{11m_1^3}{12} + \frac{11m_1^2}{8} - \frac{7m_1}{12} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{2,M}(1) &= \frac{q^{2m_2 - 2 \sum_j j m_{2j}}}{2^{m_2} \prod_j m_{2j}!} \left( 1 - m_2 q^{-1} + \left( -\frac{m_2^2}{2} + \frac{3m_2}{2} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{3,M}(1) &= \frac{q^{3m_3 - 3 \sum_j j m_{3j}}}{3^{m_3} \prod_j m_{3j}!} (1 - m_3 q^{-2} + O(q^{-3})), \\
\phi_{4,M}(1) &= \frac{q^{4m_4 - 4 \sum_j j m_{4j}}}{4^{m_4} \prod_j m_{4j}!} (1 - m_4 q^{-2} + O(q^{-3})), \\
\phi_{i,M}(1) &= \frac{q^{im_i - i \sum_j j m_{ij}}}{i^{m_i} \prod_j m_{ij}!} (1 + O(q^{-3})) \quad \text{for } i \geq 5.
\end{aligned}$$

*Proof.* Since  $N(i, q) = N^+(i, q)$  for  $i \geq 2$  it follows that  $\phi_{i,M}(1) = \phi_{i,M}^+(1)$  for  $i \geq 2$  so the expansions are as in Lemma 4.15. Thus we only need to consider  $i = 1$  for this proof.

From the definition of  $\phi_{i,M}(t)$  in Theorem 4.27,

$$\phi_{1,M}(1) = m_1! \binom{N(1, q)}{m_1} \prod_j \frac{q^{-j m_{1j}}}{m_{1j}! (1 - q^{-1} + q^{-2})^{m_{1j}}} = \frac{q}{q - m_1} \times \phi_{1,M}^+(1).$$

Firstly

$$\frac{q}{q - m_1} = \frac{1}{1 - m_1 q^{-1}} = 1 + m_1 q^{-1} + m_1^2 q^{-2} + O(q^{-3}).$$

Then from Lemma 4.15 we have that

$$\begin{aligned}
\phi_{1,M}^+(1) &= \frac{q^{m_1 - \sum_j j m_{1j}}}{\prod_j m_{1j}!} \left( 1 + \left( -\frac{m_1^2}{2} + \frac{m_1}{2} \right) q^{-1} \right. \\
&\quad \left. + \left( \frac{m_1^4}{8} - \frac{5m_1^3}{12} - \frac{m_1^2}{8} - \frac{7m_1}{12} \right) q^{-2} + O(q^{-3}) \right).
\end{aligned}$$

Multiplying together these two parts and collecting terms gives us  $\phi_{1,M}(1)$  equal to

$$\begin{aligned}
&\frac{q^{m_1 - \sum_j j m_{1j}}}{\prod_j m_{1j}!} \times \\
&\left( 1 + \left( -\frac{m_1^2}{2} + \frac{3m_1}{2} \right) q^{-1} + \left( \frac{m_1^4}{8} - \frac{11m_1^3}{12} + \frac{11m_1^2}{8} - \frac{7m_1}{12} \right) q^{-2} + O(q^{-3}) \right).
\end{aligned}$$

□

**Corollary 4.32.** *For  $M \in \mathcal{M}_{part}(r)$  we have*

$$\begin{aligned}
\phi_{1,M}(1) &= \frac{1}{m_{11}!} \left( 1 + \left( \frac{-m_{11}^2}{2} + \frac{3m_{11}}{2} \right) q^{-1} + \right. \\
&\quad \left. \left( \frac{m_{11}^4}{8} - \frac{11m_{11}^3}{12} + \frac{11m_{11}^2}{8} - \frac{7m_{11}}{12} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{2,M}(1) &= \frac{1}{2^{m_{21}} m_{21}!} \left( 1 - m_{21} q^{-1} + \left( -\frac{m_{21}^2}{2} + \frac{3m_{21}}{2} \right) q^{-2} + O(q^{-3}) \right), \\
\phi_{3,M}(1) &= \frac{1}{3^{m_{31}} m_{31}!} (1 - m_{31} q^{-2} + O(q^{-3})), \\
\phi_{4,M}(1) &= \frac{1}{4^{m_{41}} m_{41}!} (1 - m_{41} q^{-2} + O(q^{-3})), \\
\phi_{i,M}(1) &= \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-3})) \text{ for } i \geq 5.
\end{aligned}$$

*Proof.* For  $M \in \mathcal{M}_{part}(r)$  we have  $m_{ij} = 0$  for all  $i$  and  $j \geq 2$ . Hence  $im_i - i \sum_j j m_{ij} = 0$ . Thus the first term of  $\phi_{i,M}(1)$  is a constant for all  $i$  and the remaining terms are as in Lemma 4.31 □

The next corollary follows immediately from Corollary 4.18.

**Corollary 4.33.** *The leading term of  $\phi_{i,M}(1)$  is a constant for all  $i$  if and only if  $(m_{ij}) \in \mathcal{M}_{part}(r)$ .*

Now we move onto the main theorem of the section which gives the limiting proportion of cyclic matrices in  $M(V)_U$ . Theorem 1.1 for matrix algebras follows from Theorem 4.30 and this result Theorem 4.34.

**Theorem 4.34.** *For  $r \in \mathbb{Z}^+$ ,  $\lim_{n \rightarrow \infty} c_{M,r}(n) = 1 - q^{-2} + O(q^{-3})$ .*

*Proof.* By Theorem 4.30,

$$\lim_{n \rightarrow \infty} c_{M,r}(n) = \lim_{n \rightarrow \infty} \omega_r(n) \times \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \times \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1). \quad (31)$$

We calculate the expansions of each of the three parts above. From Lemma 4.24, we have that  $\lim_{n \rightarrow \infty} \omega_r(n) = \prod_{i=1}^r (1 - q^{-i}) \prod_{i=1}^{\infty} (1 - q^{-i})$  and its expansion is given by

$$\lim_{n \rightarrow \infty} \omega_r(n) = \begin{cases} 1 - 2q^{-1} + O(q^{-3}) & \text{if } r = 1 \\ 1 - 2q^{-1} - q^{-2} + O(q^{-3}) & \text{if } r \geq 2 \end{cases}$$

For the second part, a simple expansion gives

$$\frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} = 1 + q^{-1} + 2q^{-2} + O(q^{-3}).$$

Calculating the expansion of the third part,  $\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1)$ , requires more work. We will calculate the constant term in the expansion followed by the coefficient of  $q^{-1}$  and then the coefficient of  $q^{-2}$ .

**Constant Term:**

By Corollary 4.33, the leading term of  $\phi_{i,M}(1)$  is a constant if and only if  $M \in \mathcal{M}_{part}(r)$ . From Corollary 4.32, we can see that this leading term is  $\frac{1}{i^{m_{i1}} m_{i1}!}$  if  $M \in \mathcal{M}_{part}(r)$  so the constant term in  $\lim_{n \rightarrow \infty} c_{GL,r}(n)$  is

$$\sum_{M \in \mathcal{M}_{part}(r)} \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}.$$

By Corollary 4.20 this equals 1 for all  $r$ . Hence 1 is the constant term for all  $r$ .

**Coefficient of  $q^{-1}$**

For the coefficient of  $q^{-1}$ , we first sum over all  $M \in \mathcal{M}_{part}(r)$  and then look at those  $M \notin \mathcal{M}_{part}(r)$  that also contribute to the  $q^{-1}$  term.

Let  $M = (m_{ij}) \in \mathcal{M}_{part}(r)$ . By Corollary 4.32 there are exactly two ways to produce a  $q^{-1}$  term. The first is to multiply the  $q^{-1}$  term in  $\phi_{1,M}(1)$  with the constant term from each of the remaining  $\phi_{i,M}(1)$ . The second way is to multiply the  $q^{-1}$  term in  $\phi_{2,M}(1)$  with the constant term in each of the remaining  $\phi_{i,M}(1)$ .

The first way produces  $\left(-\frac{m_{11}^2}{2} + \frac{3m_{11}}{2}\right) \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  as the coefficient of  $q^{-1}$ . Summing this over all  $M \in \mathcal{M}_{part}(r)$ , gives the generating function for the contribution to the coefficient of  $q^{-1}$  as

$$-\frac{1}{2} \left( \sum_{m \geq 0} \frac{m^2 t^m}{m!} \right) \frac{e^{-t}}{1-t} + \frac{3}{2} \left( \sum_{m \geq 0} \frac{m t^m}{m!} \right) \frac{e^{-t}}{1-t}$$

by Lemma 4.19, using  $f_1(m) = m^2$  in the first case and  $f_1(m) = m$  in the second case. From the result in Lemma 4.2 the generating function for the coefficient of  $q^{-1}$  arising in this way is

$$\left( -\frac{1}{2}(t^2 + t)e^t + \frac{3}{2}te^t \right) \frac{e^{-t}}{1-t} = \left( \frac{-t^2}{2} + t \right) \frac{1}{1-t}.$$

The second way an  $M \in \mathcal{M}_{part}(r)$  can produce a  $q^{-1}$  term, produces  $-m_{21} \prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  as the coefficient of  $q^{-1}$ . Summing this over all  $M \in \mathcal{M}_{part}(r)$ , gives the generating function for the contribution to the coefficient of  $q^{-1}$  as

$$\left( - \sum_{m \geq 0} \frac{m t^{2m}}{2^m m!} \right) \frac{e^{-\frac{t^2}{2}}}{1-t}$$



by Lemma 4.19, using  $f_2(m) = -m$ . By Lemma 4.2 with  $b = 1$  and  $k = 2$ , it can be shown that  $\sum_{m \geq 0} \frac{mt^{2m}}{2^m m!} = \frac{t^2}{2} e^{\frac{t^2}{2}}$ , so the generating function for the coefficient of  $q^{-1}$  arising in this way is

$$\left( \frac{-t^2}{2} e^{\frac{t^2}{2}} \right) \frac{e^{-\frac{t^2}{2}}}{1-t} = \left( \frac{-t^2}{2} \right) \frac{1}{1-t}.$$

Summing together these two functions gives us the generating function for the coefficient of  $q^{-1}$  due to  $M \in \mathcal{M}_{part}(r)$ . This is  $\frac{-t^2+t}{1-t}$ .

Now we need to look at those  $M = (m_{ij}) \notin \mathcal{M}_{part}(r)$  that also contribute to the  $q^{-1}$  term. For  $M$  to contribute to the  $q^{-1}$  term there must exist an  $i$  such that the leading term of  $\phi_{i,M}(1)$  is not a constant and this leading term is  $q^{-1}$ . Thus  $i \sum_j m_{ij} - i \sum_j j m_{ij} = -1$ . Rewriting the equation we see that we need  $\sum_j (1-j)m_{ij} = \frac{-1}{i}$ . Since the left hand side is an integer, this equality can only hold for  $i = 1$ . Thus we need  $\phi_{1,M}(1)$  for  $i \geq 2$  to have constant leading term and  $\sum_j (1-j)m_{1j} = -1$ . By Lemma 4.17 (since  $\phi_{i,M}(1) = \phi_{i,M}^+(1)$  for  $i \geq 2$ ) the first condition implies that  $m_{ij} = 0$  for  $i \geq 2$  and  $j \geq 2$ . The second condition implies that  $m_{12} = 1$  and  $m_{1j} = 0$  for  $j \geq 3$ . Hence if  $(m_{ij}) \notin \mathcal{M}_{part}(r)$  and contributes to the  $q^{-1}$  term then

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 1) \\ m_{12} = 1 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2). \end{cases} \quad (32)$$

Note that there are no such  $(m_{ij})$  when  $r = 1$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r-2)$ . So we will think of each  $(m_{ij})$  as in (32) as a partition of  $r-2$  with  $m_{12} = 1$  appended to it.

We can see from Lemma 4.31, that for each  $M = (m_{ij})$  as in (32), for  $i \geq 2$ ,  $\phi_{i,M}(1)$  has constant term  $\frac{1}{i^{m_{i1}} m_{i1}!}$  and for  $i = 1$  the coefficient of  $q^{-1}$  is  $\frac{1}{m_{11}!}$ . Hence each of these  $(m_{ij})$  produces  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  as the coefficient of  $q^{-1}$ . Summing this over all  $M \notin \mathcal{M}_{part}(r)$  that contribute to the  $q^{-1}$  term is then the same as summing over all  $M' \in \mathcal{M}_{part}(r-2)$ . By Lemma 4.19, the generating function for the sum of  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  over all partitions of  $r$  is  $\frac{1}{1-t}$ . The generating function for the sum of  $\prod_i \frac{1}{i^{m_{i1}} m_{i1}!}$  over all partitions of  $r-2$  has no terms of degree less than 2 and so is  $\frac{t^2}{1-t}$ . Hence the generating function for the coefficient of  $q^{-1}$  due to  $M \notin \mathcal{M}_{part}(r)$  is  $\frac{t^2}{1-t}$ .

Adding the generating functions for the coefficient of  $q^{-1}$  due to  $M \in \mathcal{M}_{part}(r)$  and  $M \notin \mathcal{M}_{part}(r)$  gives us  $\frac{t}{1-t} = t + t^2 + t^3 + \dots$ . Hence for all  $r \geq 1$  the  $q^{-1}$  term in the expansion is 1.

### Coefficient of $q^{-2}$

We take the same approach in calculating the coefficient of  $q^{-2}$ . We first sum over all  $M \in \mathcal{M}_{part}(r)$ . Let  $M \in \mathcal{M}_{part}(r)$ . From the expansion of the

$\phi_{1,M}(1)$	$\phi_{2,M}(1)$	$\phi_{3,M}(1)$	$\phi_{4,M}(1)$	Generating Function
1	1	1	$-m_{41}q^{-2}$	$\left(-\frac{t^4}{4}\right)\frac{1}{1-t}$
1	1	$-m_{31}q^{-2}$	1	$\left(-\frac{t^3}{3}\right)\frac{1}{1-t}$
1	$\frac{3m_{21}}{2}q^{-2}$	1	1	$\left(\frac{3t^2}{4}\right)\frac{1}{1-t}$
1	$-\frac{m_{21}^2}{2}q^{-2}$	1	1	$\left(-\frac{t^4}{8} - \frac{t^2}{4}\right)\frac{1}{1-t}$
$-\frac{m_{11}^2}{2}q^{-1}$	$-m_{21}q^{-1}$	1	1	$\left(\frac{t^4}{4} + \frac{t^3}{4}\right)\frac{1}{1-t}$
$\frac{3m_{11}}{2}q^{-1}$	$-m_{21}q^{-1}$	1	1	$\left(-\frac{3t^3}{4}\right)\frac{1}{1-t}$
$-\frac{7m_{11}}{12}q^{-2}$	1	1	1	$\left(-\frac{7t}{12}\right)\frac{1}{1-t}$
$\frac{11m_{11}^2}{8}q^{-2}$	1	1	1	$\left(\frac{11(t^2+t)}{8}\right)\frac{1}{1-t}$
$-\frac{11m_{11}^3}{12}q^{-2}$	1	1	1	$\left(\frac{-11(t^3+3t^2+t)}{12}\right)\frac{1}{1-t}$
$\frac{m_{11}^4}{8}q^{-2}$	1	1	1	$\left(\frac{t^4+6t^3+7t^2+t}{8}\right)\frac{1}{1-t}$
Total				$\left(-t^3\right)\frac{1}{1-t}$

Table 3: Producing the generating function for the coefficient of  $q^{-2}$  in the expansion of  $\sum_M \prod_i \phi_{i,M}(1)$  due to  $M \in \mathcal{M}_{part}(r)$ .

$\phi_{i,M}(1)$  in Corollary 4.32 we can see that there are several ways to form a  $q^{-2}$  term. Each of these ways corresponds to a line of Table 3. For each line in Table 3, the contribution from  $\phi_{i,M}$  from  $i \geq 5$  is 1. If a line contains exactly one non-1 entry then that entry is a certain function of  $m_{k1}$  for some  $k$ , say  $f_k(m_{k1})$  and this line corresponds to a summand  $a_r$  of the coefficient of  $q^{-2}$ , where  $a_r$  is as given in Lemma 4.19. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemmas 4.19 and Lemma 4.2 and recorded in the last entry for this line. There are two further lines that contain two non-1 entries. These are certain functions  $f_k(m_{k1})$  with  $k = 1$  and  $f_\ell(m_{\ell 1})$  with  $\ell = 2$  and correspond to a summand  $a_r$  of the coefficient of  $q^{-2}$ , where  $a_r$  is as given in Lemma 4.21. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemmas 4.21 and Lemma 4.2 and recorded in the last entry for this line.

The sum of all the generating functions is the generating function for the coefficient of  $q^{-2}$  due to  $M \in \mathcal{M}_{part}(r)$ . It is  $\frac{-t^3}{1-t}$ .

We now need to look at the  $M \notin \mathcal{M}_{part}(r)$  that can produce a  $q^{-2}$  term. For any  $M \notin \mathcal{M}_{part}(r)$  the leading term of  $\phi_{i,M}^+(1)$  is the same as  $\varphi_{i,M}^+(1)$  for all  $i$ , so by the same reasoning as in the proof of Theorem 4.22, there are only four types of  $M \notin \mathcal{M}_{part}(r)$  that can produce a  $q^{-2}$  term. They are given below.

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 0) \\ m_{22} = 1 \\ m_{2j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \neq 2, j \geq 2) \end{cases} \quad (33)$$

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 0) \\ m_{13} = 1 \\ m_{1j} = 0 \ (j \neq 1, 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2) \end{cases} \quad (34)$$

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 0) \\ m_{12} = 2 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2) \end{cases} \quad (35)$$

and

$$(m_{ij}) = \begin{cases} m_{i1} = \text{anything } (i \geq 0) \\ m_{12} = 1 \\ m_{1j} = 0 \ (j \geq 3) \\ m_{ij} = 0 \ (i \geq 2, j \geq 2). \end{cases} \quad (36)$$

Note that there are no such  $(m_{ij})$  for  $r < 4$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (2, 2)$  and  $m'_{22} = 0$  then  $M' \in \mathcal{M}_{part}(r-4)$ . Thus we think of each  $(m_{ij})$  as in (33) as a partition of  $r-4$  with  $m_{22} = 1$  appended to it.

For  $(m_{ij})$  as in (33), we have  $m_2 = m_{21} + 1$  and  $m_i = m_{i1}$  for  $i \neq 2$ . By Lemma 4.15 we produce  $\phi_{i,M}(1)$  as follows.

$$\begin{aligned} \phi_{1,M}(1) &= \frac{1}{m_{11}!} (1 + O(q^{-1})) \\ \phi_{2,M}(1) &= \frac{q^{-2}}{2^{m_{21}} m_{21}!} \left(\frac{1}{2}\right) (1 + O(q^{-1})) \end{aligned}$$

and for  $i \geq 3$

$$\phi_{i,M}(1) = \frac{1}{i^{m_{i1}} m_{i1}!} (1 + O(q^{-1})).$$

The ' $\frac{1}{2}$ ' inside the  $\phi_{2,M}(1)$  occurs because  $2^{m_2} = 2^{m_{21}} 2^1$ .

There is only one way to make up a  $q^{-2}$  term and the first line in Table 4 corresponds to forming a  $q^{-2}$  term in this way.

Note that there are no such  $(m_{ij})$  as in (34) for  $r < 3$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 3)$  and  $m'_{13} = 0$  then  $M' \in \mathcal{M}_{part}(r-3)$ . So we think of each  $(m_{ij})$  as in (33) as a partition of  $r-3$  with  $m_{13} = 1$  appended to it.

Also note that there are no such  $(m_{ij})$  as in (35) for  $r < 4$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r-4)$ . So we think of each  $(m_{ij})$  as in (35) as a partition of  $r-4$  with  $m_{12} = 2$  appended to it.

For  $(m_{ij})$  as in (34), we have  $m_1 = m_{11} + 1$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we produce  $\phi_{i,M}(1)$  as follows.

$$\phi_{1,M}(1) = \frac{q^{-2}}{m_{11}!}(1 + O(q^{-1}))$$

and for  $i \geq 2$

$$\phi_{i,M}(1) = \frac{1}{i^{m_{i1}}m_{i1}!}(1 + O(q^{-1})).$$

There is only one way to make up a  $q^{-2}$  term and the second line in Table 4 corresponds to forming a  $q^{-2}$  term in this way.

For  $(m_{ij})$  as in (35), we have  $m_1 = m_{11} + 2$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we produce  $\phi_{i,M}(1)$  as follows.

$$\phi_{1,M}(1) = \frac{q^{-2}}{m_{11}!2!}(1 + O(q^{-1}))$$

and for  $i \geq 2$

$$\phi_{i,M}(1) = \frac{1}{i^{m_{i1}}m_{i1}!}(1 + O(q^{-1})).$$

There is again only one way to make a  $q^{-2}$  term here and the third line in Table 4 corresponds to forming a  $q^{-2}$  term in this way. Note that  $\prod_j m_{1j}! = m_{1j}!2!$ , so we can take the half out as a factor.

Note that there are no such  $(m_{ij})$  for  $r < 2$ . If we take such an  $(m_{ij}) \in \mathcal{M}(r)$  and form a new  $M' = (m'_{ij})$  with  $m'_{ij} = m_{ij}$  for all  $(i, j) \neq (1, 2)$  and  $m'_{12} = 0$  then  $M' \in \mathcal{M}_{part}(r - 2)$ . So we think of each  $(m_{ij})$  as in (36) as a partition of  $r - 2$  with  $m_{12} = 1$  appended to it.

For  $(m_{ij})$  as in (36), we have  $m_1 = m_{11} + 1$  and  $m_i = m_{i1}$  for  $i \neq 1$ , and by Lemma 4.15 we produce  $\phi_{i,M}(1)$  as follows.

$$\phi_{1,M}(1) = \frac{q^{-1}}{m_{11}!} \left( 1 + \left( -\frac{m_{11}^2}{2} + \frac{m_{11}}{2} + 1 \right) q^{-1} + O(q^{-2}) \right)$$

$$\phi_{2,M}(1) = \frac{1}{2^{m_{21}}m_{21}!} (1 - m_{21}q^{-1} + O(q^{-2}))$$

and for  $i \geq 3$

$$\phi_{i,M}(1) = \frac{1}{i^{m_{i1}}m_{i1}!} (1 + O(q^{-1})).$$

We have that  $\phi_{1,M}(1)$  has leading term involving  $q^{-1}$  so we can choose the  $q^{-1}$  term from both  $\phi_{1,M}(1)$  and  $\phi_{2,M}(1)$  or the  $q^{-2}$  term from  $\phi_{1,M}(1)$  and the constant term from  $\phi_{2,M}(1)$ . Clearly we must choose the constant term from each of  $\phi_{i,M}(1)$  for  $i \geq 3$ . The former choice corresponds to line four of Table 4 while the latter choice corresponds to lines five, six and seven of the table.

The first column of Table 4 indicates the type of array  $M$  used to produce the  $\phi_{i,M}(1)$ . If a line contains exactly one non-1 entry then that entry is a certain function of  $m_{k1}$  for some  $k$ , say  $f_k(m_{k1})$  and this line corresponds to a summand

$M$ as in:	$\phi_{1,M}(1)$	$\phi_{2,M}(1)$	$\phi_{3+,M}(1)$	Generating Function
(33)	1	$\frac{1}{2}$	1	$\left(\frac{t^4}{2}\right) \frac{1}{1-t}$
(34)	1	1	1	$\left(t^3\right) \frac{1}{1-t}$
(35)	$\frac{1}{2}$	1	1	$\left(\frac{t^4}{2}\right) \frac{1}{1-t}$
(36)	1	$-m_{21}$	1	$\left(\frac{-t^4}{2}\right) \frac{1}{1-t}$
(36)	$-\frac{m_{11}^2}{2}$	1	1	$\left(-\frac{t^3}{2} - \frac{t^4}{2}\right) \frac{1}{1-t}$
(36)	$\frac{m_{11}}{2}$	1	1	$\left(\frac{t^3}{2}\right) \frac{1}{1-t}$
(36)	1	1	1	$\left(t^2\right) \frac{1}{1-t}$
	Total			$\left(t^3 + t^2\right) \frac{1}{1-t}$

Table 4: Producing the generating function for the coefficient of  $q^{-2}$  in the expansion of  $\sum_M \prod_i \phi_{i,M}(1)$  due to  $M \in \mathcal{M}(r)$  corresponding to nonpartitions.

$a_r$  of the coefficient of  $q^{-2}$  as given in Lemma 4.19. The sum  $1 + \sum_{r=1}^{\infty} a_r t^r$  corresponding to these values of  $a_r$  is evaluated using Lemmas 4.19 and Lemma 4.2 and recorded in the last entry for this line. There are lines in which all the entries are constant and we produce the generating function by Corollary 4.20 in these cases. Each generating function has been multiplied by  $t^b$  for the positive integer  $b$  such that the arrays  $(m_{ij})$  in question correspond to partitions of  $r - b$ .

Adding all the generating functions together for those types of  $M \notin \mathcal{M}_{part}(r)$  gives  $\frac{t^3+t^2}{1-t}$ . So the generating function for the coefficient of  $q^{-2}$  is  $\frac{t^2}{1-t} = t^2 + t^3 + \dots$ . Hence

$$\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1) = \begin{cases} 1 + q^{-1} + O(q^{-3}) & \text{if } r = 1 \\ 1 + q^{-1} + q^{-2} + O(q^{-3}) & \text{if } r \geq 2 \end{cases}$$

Now that we have the expansions for all three parts we can multiply them together as in Equation 31.

When  $r = 1$  we have

$$\lim_{n \rightarrow \infty} \omega_r(n) = 1 - 2q^{-1} + O(q^{-3}),$$

$$\frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} = 1 + q^{-1} + 2q^{-2} + O(q^{-3}),$$

$$\sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1) = 1 + q^{-1} + O(q^{-3})$$

and hence

$$\lim_{n \rightarrow \infty} c_{M,r}(n) = 1 - q^{-2} + O(q^{-3}).$$

When  $r \geq 2$  we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \omega_r(n) &= 1 - 2q^{-1} - q^{-2} + O(q^{-3}), \\ \frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} &= 1 + q^{-1} + 2q^{-2} + O(q^{-3}), \\ \sum_{M \in \mathcal{M}(r)} \prod_{i=1}^r \phi_{i,M}(1) &= 1 + q^{-1} + q^{-2} + O(q^{-3})\end{aligned}$$

and hence

$$\lim_{n \rightarrow \infty} c_{M,r}(n) = 1 - q^{-2} + O(q^{-3}).$$

So for all  $r \geq 1$  we have the result.  $\square$

It was again a basic assumption of the section that  $r$ , the dimension of the invariant subspace, was greater than or equal to 1. However, we could have relaxed that assumption, allowed  $r$  to be zero and worked through the section in the same manner.

If we had allowed  $r$  to equal zero then Theorem 4.27 would state that  $C_{M,r}(t) = C_M(t)$  when  $r = 0$ , since the only matrix in  $\mathcal{M}(0)$  is the matrix containing only zeroes. Also, in the proof of Theorem 4.34, the answer for the limit of  $c_{M,0}(n)$  would be  $1 - q^{-3} + O(q^{-4})$ .

These are exactly the answers that should arise, because when  $r = 0$  the invariant subspace has dimension 0 and hence the matrix group  $\mathrm{GL}(V)_U$  is equal to  $\mathrm{GL}(V)$ . By Theorem 3.7 the limiting proportion of cyclic matrices in  $\mathrm{GL}(V)$  is indeed  $1 - q^{-3} + O(q^{-4})$ .

#### 4.4 Explicit Generating Functions For Small $r$

By Theorem 4.11 and Theorem 4.27 we can give the generating functions  $C_{\mathrm{GL},r}(t)$  and  $C_{M,r}(t)$ , for any  $r$ , in terms of Wall's generating functions  $C_{\mathrm{GL}}(t)$  and  $C_M(t)$  respectively, and functions produced by certain procedures. Using Theorem 4.14 and Theorem 4.30 we obtain algorithmically the exact limiting proportion of cyclic matrices in  $\mathrm{GL}(V)_U$  and  $M(V)_U$  respectively, for any  $r$ . For small  $r$  it is possible to calculate explicitly the generating functions as well as their associated limiting proportions.

For  $r = 1$ , we derive from Theorem 4.11 that

$$C_{\mathrm{GL},1}(t) = C_{\mathrm{GL}}(t) \times \frac{(1 - q^{-1})t}{1 - q^{-1} + tq^{-2}}.$$

This result was obtained by Jason Fulman [7, Theorem 14] as was the result below determining  $\lim_{n \rightarrow \infty} c_{\mathrm{GL},1}(n)$  [7, Corollary 2]. However Fulman derived them in the context of proportions inside a maximal parabolic subgroup of the larger general linear group  $\mathrm{GL}(n + 1, q)$ .

For matrix algebras with  $r = 1$  we find

$$\begin{aligned} C_{M,1}(t) &= C_M(t) \times q \times \frac{tq^{-1}}{1 - q^{-1} + tq^{-2}} \\ &= C_M(t) \times \frac{t}{1 - q^{-1} + tq^{-2}}. \end{aligned}$$

By Lemma 3.3, multiplying the generating functions by  $(1-t)$  and evaluating at  $t = 1$  will give us the limit of the coefficients. By recalling that  $(1-t)C_{GL}(t)$  evaluated at  $t = 1$  is  $\frac{1-q^{-5}}{1+q^{-3}}$ , we find that  $\lim_{n \rightarrow \infty} c_{GL,1}(n)$  equals

$$\frac{1 - q^{-5}}{1 + q^{-3}} \left( \frac{1 - q^{-1}}{1 - q^{-1} + q^{-2}} \right) = 1 - q^{-2} - 2q^{-3} + q^{-5} + 3q^{-6} + O(q^{-7}).$$

Then, recalling that  $(1-t)C_M(t)$  evaluated at  $t = 1$  is  $\frac{1-q^{-5}}{(1-q^{-1})(1-q^{-2})}$  and that  $\lim_{n \rightarrow \infty} \omega_1(n) = (1 - q^{-1}) \prod_{i=1}^{\infty} (1 - q^{-i})$  we find that  $\lim_{n \rightarrow \infty} c_{M,1}(n)$  equals

$$\begin{aligned} &\frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \times \frac{1}{1 - q^{-1} + q^{-2}} \times (1 - q^{-1}) \prod_{i=1}^{\infty} (1 - q^{-i}) \\ &= 1 - q^{-2} - 2q^{-3} - q^{-4} + 2q^{-6} + O(q^{-7}). \end{aligned}$$

When  $r = 2$ , we find that  $\mathcal{M}(2)$  has three elements. They are

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which correspond to two distinct monic degree 1 irreducibles of multiplicity 1, one monic degree 1 irreducible of multiplicity 2, and one monic degree 2 irreducible of multiplicity 1, respectively.

For each element  $M$  in  $\mathcal{M}(2)$  we calculate  $\prod_i \phi_{i,M}^+(t)$  and then sum over  $M$  and multiply by  $C_{GL}(t)$  to give  $C_{GL,2}(t)$ . Similarly for each element  $M$  of  $\mathcal{M}(2)$  we calculate  $\prod_i \phi_{i,M}(t)$  and then sum over  $M$  and multiply by  $C_M(t)$  to give  $C_{M,2}(t)$ . This yields that  $C_{GL,2}(t)$  is equal to

$$C_{GL}(t) \left( \frac{t^2(1 - q^{-1})(1 - 2q^{-1})}{2(1 - q^{-1} + tq^{-2})^2} + \frac{t^2(q^{-1} - q^{-2})}{1 - q^{-1} + tq^{-2}} + \frac{t^2(1 - q^{-1})}{2(1 - q^{-2} + t^2q^{-4})} \right)$$

and  $C_{M,2}(t)$  equal to

$$C_M(t) \left( \frac{t^2(1 - q^{-1})}{2(1 - q^{-1} + tq^{-2})^2} + \frac{t^2q^{-1}}{1 - q^{-1} + tq^{-2}} + \frac{t^2(1 - q^{-1})}{2(1 - q^{-2} + t^2q^{-4})} \right).$$

Applying Lemma 3.3, we find that the limiting proportions satisfy  $c_{GL,2}(\infty)$  equal to

$$= \frac{1 - q^{-5}}{1 + q^{-3}} \left( \frac{(1 - q^{-1})(1 - 2q^{-1})}{2(1 - q^{-1} + q^{-2})^2} + \frac{(q^{-1} - q^{-2})}{1 - q^{-1} + q^{-2}} + \frac{(1 - q^{-1})}{2(1 - q^{-2} + q^{-4})} \right) \\ = 1 - q^{-2} - 3q^{-3} + q^{-4} + 3q^{-5} + 4q^{-6} + O(q^{-7}).$$

and  $\lim_{n \rightarrow \infty} \frac{c_{M,2}(n)}{\omega_2(n)}$  is equal to

$$\frac{1 - q^{-5}}{(1 - q^{-1})(1 - q^{-2})} \left( \frac{(1 - q^{-1})}{2(1 - q^{-1} + q^{-2})^2} + \frac{q^{-1}}{1 - q^{-1} + q^{-2}} + \frac{(1 - q^{-1})}{2(1 - q^{-2} + q^{-4})} \right) \\ = 1 + 2q^{-1} + 4q^{-2} + 3q^{-3} + 3q^{-4} + q^{-5} + 2q^{-6} + O(q^{-7})$$

giving

$$\lim_{n \rightarrow \infty} c_{M,2}(n) = 1 - q^{-2} - 4q^{-3} - q^{-4} + 4q^{-5} + 5q^{-6} + O(q^{-7}).$$

**Remark 4.35.** Equation (37) and Equation (38) summarise the results of  $\lim_{n \rightarrow \infty} c_{GL,r}(n)$  and  $\lim_{n \rightarrow \infty} c_{M,r}(n)$  respectively for  $r = 1, 2$ . They include similar calculations performed in Mathematica [13] for other small values of  $r$ .

$$c_{GL,r}(\infty) = \lim_{n \rightarrow \infty} c_{GL,r}(n) \\ = \begin{cases} 1 - q^{-2} - 2q^{-3} + q^{-5} + 3q^{-6} + q^{-7} + O(q^{-8}) & \text{for } r = 1, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 3q^{-5} + 4q^{-6} - 2q^{-7} + O(q^{-8}) & \text{for } r = 2, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 4q^{-5} + 4q^{-6} - 5q^{-7} + O(q^{-8}) & \text{for } r = 3, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 4q^{-5} + 4q^{-6} - 6q^{-7} + O(q^{-8}) & \text{for } r = 4, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 4q^{-5} + 4q^{-6} - 6q^{-7} + O(q^{-8}) & \text{for } r = 5, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 4q^{-5} + 4q^{-6} - 6q^{-7} + O(q^{-8}) & \text{for } r = 6, \\ 1 - q^{-2} - 3q^{-3} + q^{-4} + 4q^{-5} + 4q^{-6} - 6q^{-7} + O(q^{-8}) & \text{for } r = 7. \end{cases} \quad (37)$$

$$c_{M,r}(\infty) = \lim_{n \rightarrow \infty} c_{M,r}(n) \\ = \begin{cases} 1 - q^{-2} - 2q^{-3} - q^{-4} + 2q^{-6} + 3q^{-7} + O(q^{-8}) & \text{for } r = 1, \\ 1 - q^{-2} - 4q^{-3} - q^{-4} + 4q^{-5} + 5q^{-6} + 4q^{-7} + O(q^{-8}) & \text{for } r = 2, \\ 1 - q^{-2} - 4q^{-3} - 3q^{-4} + 4q^{-5} + 11q^{-6} + 8q^{-7} + O(q^{-8}) & \text{for } r = 3, \\ 1 - q^{-2} - 4q^{-3} - 3q^{-4} + 2q^{-5} + 11q^{-6} + 14q^{-7} + O(q^{-8}) & \text{for } r = 4, \\ 1 - q^{-2} - 4q^{-3} - 3q^{-4} + 2q^{-5} + 9q^{-6} + 14q^{-7} + O(q^{-8}) & \text{for } r = 5, \\ 1 - q^{-2} - 4q^{-3} - 3q^{-4} + 2q^{-5} + 9q^{-6} + 12q^{-7} + O(q^{-8}) & \text{for } r = 6, \\ 1 - q^{-2} - 4q^{-3} - 3q^{-4} + 2q^{-5} + 9q^{-6} + 12q^{-7} + O(q^{-8}) & \text{for } r = 7. \end{cases} \quad (38)$$

Our results in Equation 37 agree with Theorem 4.22 which states that  $\lim_{n \rightarrow \infty} c_{GL,r}(n)$  is  $1 - q^{-2} + O(q^{-3})$  for all  $r$ . Similarly the results in Equation 38 agree with Theorem 4.34 which states that the limit of  $c_{M,r}(n)$  is  $1 - q^{-2} + O(q^{-3})$  for all  $r$ .



Although Theorem 4.22 and Theorem 4.34 give the limiting proportion for all  $r$  this does not imply that  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  or  $\lim_{n \rightarrow \infty} c_{\text{M},r}(n)$  are independent of  $r$ . In the expansion of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$ , the  $q^{-2}$  term is independent of  $r$  but the later terms do depend on  $r$  as Equation 37 clearly shows.

We believe that for large enough  $r$  the coefficient of  $q^{-3}$  will eventually become constant. In fact we believe that this is the case for all terms. Stated formally, we believe that for all  $d \geq 0$  there exists  $r_d$  such that the coefficient of  $q^{-d}$  in the expansion of  $\lim_{n \rightarrow \infty} c_{\text{GL},r}(n)$  as a power series in  $q^{-1}$  is the same for all  $r \geq r_d$ . By Theorem 4.22 this statement is true for  $d = 0, 1, 2$  but remains to be proved for all terms in the expansion.

## 4.5 A Family of Noncyclic Matrices

By Theorem 1.1, the proportion of cyclic matrices in  $\text{GL}(V)_U$ , as the dimension of  $V$  tends to infinity, approaches  $1 - q^{-2} + O(q^{-3})$ . Hence there exists a set  $S(V)_U$  of noncyclic matrices in  $\text{GL}(V)_U$  with proportion  $\frac{|S(V)_U|}{|\text{GL}(V)_U|} = q^{-2} + O(q^{-3})$ . We construct such a set in this section.

We first choose a basis for  $U$  and extend it to be a basis for  $V$ . Let  $u_1, \dots, u_r$  be a basis for  $U$  and let  $u_1, \dots, u_r, u_{r+1}, \dots, u_n$  be a basis for  $V$ . Let  $U_0 = \langle u_2, \dots, u_r \rangle$  and let  $V_0 = \langle u_1, \dots, u_{n-1} \rangle$ . For a fixed  $\lambda \in \mathbb{F}_q$  let

$$\mathcal{T}_\lambda = \left\{ T \in \text{GL}(V)_U \left| \begin{array}{l} (t - \lambda)^3 \nmid c_T(t), \\ \exists w_1 \in u_1 + U_0 \text{ such that } w_1 T = \lambda w_1 \text{ and} \\ \exists w_2 \in u_n + V_0 \text{ such that } w_2 T = \lambda w_2 \end{array} \right. \right\}. \quad (39)$$

Each of the matrices in  $\mathcal{T}_\lambda$  is noncyclic since the  $(t - \lambda)$ -primary component is the direct sum of two cyclic summands. We prove some lemmas which will aid us in proving the upcoming theorem which ultimately exhibits the set of noncyclic matrices in  $\text{GL}(V)_U$  with proportion  $q^{-2} + O(q^{-3})$  that we are looking for.

**Lemma 4.36.** *For  $\lambda \in \mathbb{F}_q^*$ ,*

$$|\mathcal{T}_\lambda| = q^{n^2+r^2-rn-3} + O(q^{n^2+r^2-rn-4}).$$

*Proof.* We count the number of triples  $(w_1, w_2, T)$ , where  $T$  is a matrix in  $\mathcal{T}_\lambda$ , and  $w_1$  and  $w_2$  are  $\lambda$ -eigenvectors for  $T$  satisfying the criteria for  $\mathcal{T}_\lambda$  in Equation (39). Since  $w_1 \in u_1 + U_0 \subseteq U$  and  $w_2 \notin U$ , the  $\lambda$ -eigenspace of  $T$  is  $W = \langle w_1, w_2 \rangle$ ,  $W \cap U = \langle w_1 \rangle$  and  $W \cap (u_1 + U_0) = \{w_1\}$ .

Suppose  $T \in \mathcal{T}_\lambda$  and let  $W$  be the  $\lambda$ -eigenspace of  $T$ . Then  $W = \langle w_1, w_2 \rangle$  for some  $\lambda$ -eigenvectors  $w_1 \in u_1 + U_0$  and  $w_2 \in u_n + V_0$ . As explained in the previous paragraph,  $w_1$  is determined uniquely by  $T$ .

Now let  $w'_2$  be an alternative choice for the second basis vector, that is, suppose that  $w'_2 T = \lambda w'_2$  with  $w'_2 \in u_n + V_0$  and  $w'_2 \neq w_2$ . Then  $0 \neq w_2 - w'_2 \in W \cap V_0$ , and as  $w_2 \in W \setminus V_0$  and  $\dim(W) = 2$ , it follows that  $W \cap V_0 = \langle w_2 - w'_2 \rangle$ . However,  $w_1 \in W \cap U \subseteq W \cap V_0$ . Hence  $w_2 - w'_2 = aw_1$  for some  $a \in \mathbb{F}_q$ . Moreover for each  $a \in \mathbb{F}_q$ ,  $w'_2 = aw_1 + w_2$  lies both in  $W$  and in  $u_n + V_0$ . Hence

there are  $q$  choices for the second basis vector for  $W$  corresponding to the  $q$  choices for  $a$ .

We will continue the count of triples  $(w_1, w_2, T)$  but take this little break to note that  $|\mathcal{T}_\lambda|$  is equal to the number of triples  $(w_1, w_2, T)$  divided by  $q$  because each  $T$  has  $q$  possibilities for the pair  $(w_1, w_2)$ .

First we choose vectors  $w_1 \in u_1 + V_0$  and  $w_2 \in u_n + V_0$ . Note that  $w_1, u_2, \dots, u_{n-1}, w_2$  is also a basis for  $V$ . For each  $T$  occurring in a triple  $(w_1, w_2, T)$  with these chosen vectors, the matrix representing  $T$  with respect to the basis  $w_1, u_2, \dots, u_{n-1}, w_2$  must be of the form

$$\left( \begin{array}{cccc|cccc} \lambda & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ * & & & & 0 & \dots & \dots & 0 \\ \vdots & & & & 0 & \dots & \dots & 0 \\ * & & & A & 0 & \dots & \dots & 0 \\ \hline * & \dots & \dots & * & & & & * \\ * & \dots & \dots & * & & B & & \vdots \\ * & \dots & \dots & * & & & & * \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & \lambda \end{array} \right)$$

where  $*$  represents any value from the field and  $A$  and  $B$  are  $(r-1) \times (r-1)$  and  $(n-r-1) \times (n-r-1)$  invertible matrices respectively, neither of which has  $t$  nor  $t - \lambda$  as a factor of their characteristic polynomial.

It is not difficult to show that there are  $N(r) \geq q^{(r-1)^2} - O(q^{(r-1)^2-1})$  choices for the matrix  $A$  and  $N(n-r) \geq q^{(n-r-1)^2} - O(q^{(n-r-1)^2-1})$  choices for the matrix  $B$  (a detailed proof is given in [4, Lemma 2.1.5]). Then there are  $n-2+r(n-r-1)$  entries  $*$  and there are  $q^{n-2+r(n-r-1)}$  choices for these  $*$  entries. Hence there are  $N(r)N(n-r)q^{n-2+r(n-r-1)} \geq q^{n^2+r^2-nr-n-r} + O(q^{n^2+r^2-nr-n-r-1})$  such matrices  $T$  for a given  $w_1$  and  $w_2$ .

Then to finish our count of the number of triples  $(w_1, w_2, T)$  we need to multiply by the number of such  $w_1$  and  $w_2$  which is  $q^{r-1}$  and  $q^{n-1}$  respectively. Hence the number of triples  $(w_1, w_2, T)$  is  $q^{n^2+r^2-rn-2} + O(q^{n^2+r^2-rn-3})$  and dividing by  $q$  gives  $|\mathcal{T}_\lambda| = q^{n^2+r^2-rn-3} + O(q^{n^2+r^2-rn-4})$ .  $\square$

**Lemma 4.37.** For  $\lambda, \gamma \in \mathbb{F}_q^*$  with  $\lambda \neq \gamma$ ,

$$|\mathcal{T}_\lambda \cap \mathcal{T}_\gamma| \leq q^{n^2+r^2-nr-6}.$$

*Proof.* Let  $T \in \mathcal{T}_\lambda \cap \mathcal{T}_\gamma$  for  $\lambda \neq \gamma$ . Since  $T \in \mathcal{T}_\lambda$  there exists  $w_1 \in u_1 + U_0$  such that  $w_1 T = \lambda w_1$  and there exists  $w_2 \in u_n + V_0$  such that  $w_2 T = \lambda w_2$ . Similarly since  $T \in \mathcal{T}_\gamma$  there exists  $x_1 \in u_1 + U_0$  such that  $x_1 T = \gamma x_1$  and there exists  $x_2 \in u_n + V_0$  such that  $x_2 T = \gamma x_2$ . Since  $\langle w_1, w_2 \rangle, \langle x_1, x_2 \rangle$  lie in distinct eigenspaces for  $T$ ,  $\langle w_1, w_2 \rangle \cap \langle x_1, x_2 \rangle = 0$  so  $\{w_1, w_2, x_1, x_2\}$  is linearly independent.

Taking a similar approach to that of Lemma 4.36, we count the number of tuples of the form  $(w_1, w_2, x_1, x_2, T)$  and relate that back to the size of  $|\mathcal{T}_\lambda \cap \mathcal{T}_\gamma|$ .

From the proof of Lemma 4.36, we know that for a given  $T \in \mathcal{T}_\lambda \cap \mathcal{T}_\gamma$ ,  $w_1$  is uniquely determined and there are  $q$  choices for  $w_2$ . Similarly,  $x_1$  is uniquely determined and there are  $q$  choices for  $x_2$ . So  $|\mathcal{T}_\lambda \cap \mathcal{T}_\gamma|$  equals the number of tuples  $(w_1, w_2, x_1, x_2, T)$  divided by  $q^2$ .

There are  $q^{r-1}$  choices for  $w_1 \in u_1 + U_0$ . For a given  $w_1$ , since  $x_1 \in u_1 + U_0 = w_1 + U_0$ , there are  $q^{r-1} - 1$  choices for  $x_1$  and the action of  $T$  on  $\langle w_1, x_1 \rangle$  is determined uniquely. In particular,  $u_0 = x_1 - w_1 \in U$  and  $u_0 T = \gamma x_1 - \lambda w_1 = \gamma u_0 + (\gamma - \lambda)w_1$ . Similarly there are  $q^{n-1}$  choices for  $w_2 \in u_n + V_0$ , and for a given  $w_2$ , there are  $q^{n-1} - 1$  choices for  $x_2$  and the action of  $T$  on  $\langle w_1, w_2, x_1, x_2 \rangle$  is uniquely determined. In particular,  $v_0 = x_2 - w_2 \in V_0$  and  $v_0 T = \gamma x_2 - \lambda w_2 = \gamma v_0 + (\gamma - \lambda)w_2$ .

Suppose  $(w_1, x_1, w_2, x_2)$  are as above and  $T$  occurs with them in a tuple. Since  $w_1, u_0, v_0, w_2$  are linearly independent we can extend them to a basis  $w_1, u_0, y_3, \dots, y_{n-2}, v_0, w_2$  for  $V$  such that  $w_1, u_0, y_3, \dots, y_r$  is a basis for  $U$ . With respect to this basis  $T$  must be of the form

$$\left( \begin{array}{cccc|ccc} \lambda & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \gamma - \lambda & \gamma & 0 & \dots & 0 & 0 & \dots & 0 \\ * & * & & & & 0 & \dots & 0 \\ \vdots & \vdots & & & A & 0 & \dots & 0 \\ * & * & & & & 0 & \dots & 0 \\ \hline * & & \dots & & * & & * & * \\ * & & \dots & & * & B & \vdots & \vdots \\ * & & \dots & & * & & * & * \\ 0 & & \dots & & 0 & 0 & \dots & 0 & \gamma & \gamma - \lambda \\ 0 & & \dots & & 0 & 0 & \dots & 0 & 0 & \lambda \end{array} \right)$$

where  $*$  represents any value from the field and  $A$  and  $B$  are  $(r-2) \times (r-2)$  and  $(n-r-2) \times (n-r-2)$  invertible matrices respectively. We this time allow  $A$  and  $B$  to have  $t - \lambda$  and  $t - \gamma$  as factors for simplicity despite it being an over count.

There are less than  $q^{(r-2)^2}$  such matrices  $A$ , there are less than  $q^{(n-r-2)^2}$  such matrices  $B$  and there are  $q^{2n-8+r(n-r-2)}$  possible values for the  $*$  entries. When we multiply these together along with the  $q^{r-1}(q^{r-1} - 1)q^{n-1}(q^{n-1} - 1)$  choices for  $(w_1, w_2, x_1, x_2)$ , we get the number of tuples  $(w_1, w_2, x_1, x_2, T)$  to be less than  $q^{n^2+r^2-nr-4}$ . Hence, after dividing by  $q^2$ , we get

$$|\mathcal{T}_\lambda \cap \mathcal{T}_\gamma| \leq q^{n^2+r^2-nr-6}.$$

□

We now state and prove Theorem 4.38 which exhibits a family of noncyclic matrices in  $\text{GL}(V)_U$  with proportion  $q^{-2} + O(q^{-3})$ .

**Theorem 4.38.** *The union of  $\mathcal{T}_\lambda$  for  $\lambda \in \mathbb{F}_q^*$  forms a set of noncyclic matrices in  $\text{GL}(V)_U$  with limiting proportion  $q^{-2} + O(q^{-3})$  as  $n$  tends to infinity.*

*Proof.* We cannot simply take the sum of  $|\mathcal{T}_\lambda|$  over all  $\lambda$  to obtain  $|\cup \mathcal{T}_\lambda|$  since some matrices belong to more than one  $\mathcal{T}_\lambda$  but we can subtract off the number of those matrices which belong to two or more  $\mathcal{T}_\lambda$  to give

$$|\cup_{\lambda \in \mathbb{F}_q^*} \mathcal{T}_\lambda| \geq (q-1)|\mathcal{T}_\lambda| - \binom{q-1}{2} |\mathcal{T}_\lambda \cap \mathcal{T}_\gamma|$$

by the Principle of Inclusion and Exclusion. By Lemma 4.36 and Lemma 4.37,

$$|\cup_{\lambda \in \mathbb{F}_q^*} \mathcal{T}_\lambda| \geq q^{n^2+r^2-nr-2} + O(q^{n^2+r^2-nr-3}).$$

We then simply divide this by  $|\mathrm{GL}(V)_U| = q^{n^2+r^2-nr} + O(q^{n^2+r^2-nr-1})$  to give

$$\frac{|\cup_{\lambda \in \mathbb{F}_q^*} \mathcal{T}_\lambda|}{|\mathrm{GL}(V)_U|} = q^{-2} + O(q^{-3}).$$

□

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